Øksendal: Stochastic Differential Equations

Solutions Manual
Christopher Kennedy

May 3, 2021
# Contents

1. Introduction ........................................... 2  
2. Some Mathematical Preliminaries ................. 3  
3. Itô Integrals ........................................... 9  
4. The Itô Formula ....................................... 14  
5. Stochastic Differential Equations ................. 20  
6. The Filtering Problem ................................ 28  
7. Diffusions: Basic Properties ......................... 29  
8. Other Topics in Diffusion Theory ................. 30  
9. Applications to Boundary Value Problems ......... 31  
10. Applications to Optimal Stopping ................. 32  
11. Applications to Stochastic Control ............... 33  
12. Applications to Mathematical Finance .......... 34
Chapter 1

Introduction

This is a solutions manual for Stochastic Differential Equations by Bernt Øksendal. This is a working document last updated May 3, 2021. Progress to date:

- Chapter 2: Problems #1-17
- Chapter 3: Problems #1-17
- Chapter 4: Problems #1-15
- Chapter 5: Problems #1-17
- Chapters 6–12: none so far
Chapter 2

Some Mathematical Preliminaries

1. Suppose $X : \Omega \rightarrow \mathbb{R}$ is a function that assumes countably many values $\{a_j\}$ in $\mathbb{R}$.

   (a) Note that $X$ is a random variable if and only if it is measurable. If $X : \Omega \rightarrow \mathbb{R}$ is measurable, then $U = X^{-1}(\mathbb{R} \setminus a_k) \in \mathcal{F}$ and thus $X^{-1}(a_k) = \Omega \setminus U \in \mathcal{F}$, $\forall k$. On the other hand, if $X^{-1}(a_k) \in \mathcal{F}$, $\forall k$, then Borel set $V \subseteq \mathbb{R}$, $X^{-1}(V) = \bigcup_{a_k \in V} X^{-1}(a_k) \in \mathcal{F}$ and thus $X$ is measurable.

   (b) Compute $\mathbb{E}(|X|) = \int_{\mathbb{R}} |x| \, d\mathbb{P}_X = \int_{\bigcup_{k=1}^{\infty} \{a_k\}} |x| \, d\mathbb{P}_X = \sum_{k=1}^{\infty} |a_k| \mathbb{P}(X = a_k)$. 

   (c) If $\mathbb{E}(|X|) < \infty$, then the series 

   $$ \mathbb{E}(X) = \int_{\mathbb{R}} x \, d\mathbb{P}_X = \int_{\bigcup_{k=1}^{\infty} \{a_k\}} x \, d\mathbb{P}_X = \sum_{k=1}^{\infty} a_k \mathbb{P}(X = a_k) $$

   is absolutely convergent and therefore converges.

   (d) If $f$ is measurable and $|f|$ is bounded by $M$, then 

   $$ \mathbb{E}(|f(X)|) = \int_{\mathbb{R}} |f(x)| \, d\mathbb{P}_X \leq \int_{\mathbb{R}} M \, d\mathbb{P}_X = M \int_{\mathbb{R}} d\mathbb{P}_X = M < \infty. $$

   Hence,

   $$ \mathbb{E}(f(X)) = \int_{\mathbb{R}} f(x) \, d\mathbb{P}_X = \int_{\bigcup_{k=1}^{\infty} \{a_k\}} f(x) \, d\mathbb{P}_X = \sum_{k=1}^{\infty} f(a_k) \mathbb{P}(X = a_k) $$

   is absolutely convergent and therefore converges.

2. Let $F(x) = \mathbb{P}(X \leq x)$ be the distribution function of $X$.

   (a) By monotonicity of $\mathbb{P}$, $0 = \mathbb{P}(\emptyset) \leq \mathbb{P}(X \leq x) \leq \mathbb{P}(\mathbb{R}) = 1$. Now, by the Monotone Convergence Theorem, 

   $$ \lim_{n \to \infty} F(n) = \lim_{n \to \infty} \int_{\mathbb{R}} X(-\infty,n] \, d\mathbb{P}(x) = \int_{\mathbb{R}} d\mathbb{P}(x) = 1. $$
Similarly, for \( G(n) := 1 - F(-n) \), we have
\[
\lim_{n \to \infty} G(n) = \lim_{n \to \infty} \int_{\mathbb{R}} (1 - \chi(-\infty, -n]) dP_X(x) = 1.
\]

Moreover, \( F \) is increasing by monotonicity of \( P \) and finally, again by Monotone Convergence,
\[
\lim_{h \to 0^+} 1 - F(x + h) + F(x) = \lim_{h \to 0^+} \int_{\mathbb{R}} (1 - \chi(x, x + h]) d\mathbb{P}(x) = \int_{\mathbb{R}} d\mathbb{P}(x) = 1
\]
and so \( \lim_{h \to 0^+} F(x + h) = F(x) \), i.e. \( F \) is right-continuous.

(b) Compute the expectation
\[
\mathbb{E}(g(X)) = \int_{\mathbb{R}} g(x) d\mathbb{P}(x) = \int_{\mathbb{R}} g(x) \chi(-\infty, x] d\mathbb{P}(x) = \int_{\mathbb{R}} g(x) dF(x).
\]

(c) Compute the density of \( B_t^2 \)
\[
F(u) := \mathbb{P}(B_t^2 \leq u) = \mathbb{P}(-\sqrt{u} \leq B_t \leq \sqrt{u})
\]
\[
= 2 \int_{[0, \sqrt{u}]} p(y) dy
\]
\[
= 2 \int_{[0, u]} \frac{p(\sqrt{u})}{2\sqrt{u}} du
\]
\[
= \int_{(-\infty, u]} \chi_{[0, \infty]} \frac{p(\sqrt{u})}{\sqrt{u}} du.
\]
and so \( p(u) = \chi_{[0, \infty]} \frac{p(\sqrt{u})}{\sqrt{u}} \) where \( p(u) \) is the density of \( B_t \).

3. Since \( \mathcal{H}_i \) is a \( \sigma \)-algebra, \( \emptyset \in \mathcal{H}_i, \forall i \in I \). So \( \emptyset \in \mathcal{H} = \bigcap_{i \in I} \mathcal{H}_i \). If \( \{U_j\}_{j \in \mathbb{N}} \subseteq \mathcal{H}_i \) for each \( i \in I \) and so \( \Omega \setminus U_j \in \mathcal{H}_i \) and \( \bigcup_{j \in \mathbb{N}} U_j \in \mathcal{H}_i, \forall i \in I \). Conclude that \( \Omega \setminus \bigcup_{j \in \mathbb{N}} U_j \in \mathcal{H}_i \) and \( \mathcal{H} = \bigcap_{i \in I} \mathcal{H}_i \) is also a \( \sigma \)-algebra.

4. Let \( X : \Omega \mapsto \mathbb{R} \) be a random variable with \( \mathbb{E}(|X|^p) < \infty \).

(a) Let \( A = \{\omega \in \Omega \mid |X| \geq \lambda > 0\} \) and compute
\[
\mathbb{E}(|X|^p) = \int_{\Omega} |X|^p d\mathbb{P} \geq \int_A |X|^p d\mathbb{P} \geq \lambda^p \int_A d\mathbb{P} = \lambda^p \mathbb{P}(|X| \geq \lambda).
\]

(b) By Chebychew, \( \mathbb{P}(|X| \geq \lambda) = \mathbb{P}(e^{|X|} \geq e^\lambda) \leq \frac{1}{e^\lambda} \mathbb{E}(e^{k|X|}) = Me^{-k\lambda}. \)
5. Since the measures are $\sigma$-finite, $f(x, y) = xy$ is $P_X \otimes P_Y$ measurable and $\mathbb{E}(|XY|) < \infty$, apply Fubini-Tonelli and compute
\[
\mathbb{E}(XY) = \int_{\mathbb{R}^2} xy \, dP_{XY}(x, y) \\
= \int_{\mathbb{R}^2} xy \, dP_X(x) \otimes dP_Y(y) \\
= \int_{\mathbb{R}} y \left( \int_{\mathbb{R}} x \, dP_X(x) \right) dP_Y(y) \\
= \mathbb{E}(X) \int_{\mathbb{R}} y \, dP_Y(y) \\
= \mathbb{E}(X) \mathbb{E}(Y).
\]

6. (Borel-Cantelli) Let $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{F}$ and suppose $\sum_{k=1}^{\infty} \mathbb{P}(A_k) < \infty$. Then
\[
\mathbb{P}(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k) \leq \lim_{m \to \infty} \sup_{k \geq m} \mathbb{P}(A_k) = 0
\]
by dominated convergence.

7. Let $\Omega = \bigsqcup_{i=1}^{n} G_i$.

(a) Note $\emptyset \in \mathcal{G}$ and $\mathcal{G}$ is closed under unions by construction. It is also closed under complements as $\Omega \setminus G_i = \bigcup_{j \neq i} G_j \in \mathcal{G}$.

(b) Write a new sequence defined by $F_i = G_i \setminus \bigcup_{j \leq i} F_j$ and $\{F_i\}$ will satisfy (a).

(c) Note that $\{X^{-1} (x \in \mathbb{R})\} \subseteq \mathcal{F}$ is disjoint. So, by (a) and (b), $\mathcal{F}$ is finite if and only if all but finitely many $X^{-1} (x \in \mathbb{R})$ are empty.

8. Let $B_t$ be a 1-dimensional Wiener process.

(a) By Equation 2.2.3, since $B_t \sim N(0, t)$,
\[
\mathbb{E}(e^{iuB_t}) = \exp \left( -\frac{u^2}{2} \mathbb{V}(B_t) + iu \mathbb{E}(B_t) \right) = e^{-\frac{u^2 t}{2}}.
\]

(b) Comparing power series coefficients, we deduce that
\[
\frac{(iu)^{2n}}{(2n)!} \mathbb{E}(B_t^{2n}) = \frac{1}{n!} \left( -\frac{u^2 t}{2} \right)^n,
\]
and so $\mathbb{E}(B_t^{2n}) = \frac{(2n)!}{2^n n!} t^n$.
(c) Integrating by parts, compute the $n^{th}$ moment of $B_t$

$$\mathbb{E}(B_t^{2k}) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} x^{2k} e^{-\frac{x^2}{2t}} \, dx$$

$$= x^{2k-1} \frac{2t}{\pi} \int_{-\infty}^{\infty} u e^{-u^2} \, du \bigg|_{x=\infty} - \int_{\mathbb{R}} (2k-1)x^{2k-2} \frac{2t}{\pi} \int_{\mathbb{R}} u e^{-u^2} \, du$$

$$= -(2k-1) \frac{2t}{\pi} \int_{\mathbb{R}} x^{2k-2} \left( \frac{1}{2} e^{-\frac{x^2}{2t}} \right) \, dx$$

$$= (2k-1)t \cdot \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} x^{2k-2} e^{-\frac{x^2}{2t}} \, dx$$

$$= (2k-1)t \mathbb{E}(B_t^{2k-2}).$$

As $\mathbb{E}(B_t^2) = t$, we have that $\mathbb{E}(B_t^{2k}) = \frac{(2k)!t^{k-1}}{2^{k+1}k!} \cdot t = \frac{(2k)!t^k}{2^{k+1}k!}$.

(d) Check the base case, $n = 2k = 2$, where $\mathbb{E}(B_t^2) = \frac{2t}{2t} = t$. If the claim is true for $n = 2k$, then

$$\mathbb{E}(B_t^{2k+2}) = (2k-1)t \mathbb{E}(B_t^{2k}) = (2k+1)t \cdot \frac{(2k)!t^k}{2^{k+1}k!} = \frac{(2k+2)!t^{k+1}}{2^{k+1}(k+1)!},$$

and so it is also true for $n = 2(k+1) = 2k + 2$, thus completing the induction step.

9. Note that $\{X_t\}$ and $\{Y_t\}$ have the same distributions since neither distribution has any atoms and they agree except on a zero set $\forall t \geq 0$. Yet $t \mapsto X_t$ is discontinuous while $t \mapsto Y_t$ is continuous.

10. As $B_t$ is Brownian, $B_{t+h} - B_t \sim N(0, h)$. Since $h$ is fixed, $\{B_{t+h} - B_t\}_{h \geq 0}$ have the same distributions $\forall t \geq 0$.

11. As $B_0 = \left( B_0^{(1)}, B_0^{(2)}, \ldots, B_0^{(n)} \right) = 0$, $B_0^{(j)} = 0$ for all $j \in \{1, \ldots, n\}$. $B_t$ is almost surely continuous only if its components are almost surely continuous. Each component is normally distributed with $\mathbb{E}(B_t^{(j)}) = 0$ as $\mathbb{E}(B_t) = 0$ and $\text{Cov}(B_t^{(i)}, B_t^{(j)}) = t \delta_{ij}$ as $\text{Cov}(B_t) = tI$.

12. Let $W_t := B_{s+t} - B_s$ where $s \geq 0$ is fixed. Then $W_0 = B_s - B_s = 0$ and $W_t$ is almost surely continuous as the sum of two almost surely continuous stochastic processes. Noting $W_{t_2} - W_{t_1} = B_{s+t_2} - B_{s+t_1}$ is independent of both $B_{s+t_1}$ and $B_s$, deduce that $W_{t_2} - W_{t_1}$ is independent of $W_{t_1} = B_{s+t_1} - B_s$. The expected value is $\mathbb{E}(W_t) = \mathbb{E}(B_{s+t}) - \mathbb{E}(B_s) = 0$ and the variance is

$$\mathbb{V}(W_t) = \mathbb{E}((B_{s+t} - B_s)^2)$$

$$= \mathbb{E}(B_{s+t}^2) - 2\mathbb{E}(B_s B_{s+t}) + \mathbb{E}(B_s^2)$$

$$= \mathbb{E}(B_{s+t}^2) - 2\mathbb{E}(B_s(B_{s+t} - B_s)) - \mathbb{E}(B_s^2)$$

$$= \mathbb{E}(B_{s+t}^2) - 2\mathbb{E}(B_s)\mathbb{E}(B_{s+t} - B_s) - \mathbb{E}(B_s^2)$$

$$= (s + t) - 0 - s$$

$$= t.$$
Since $W_t$ is the sum of two normal distributions, it is also normal and $W_t \sim N(0, t)$.

13. Compute

$$\mathbb{P}_0(B_t \in D_\rho) = \int_{|x| < \rho} \frac{1}{2\pi t} e^{-\frac{|x|^2}{2t}} d^2x = \frac{2\pi}{2\pi t} \int_0^\rho r e^{-\frac{r^2}{2t}} dr = \int_0^{\frac{\rho^2}{2t}} e^{-u} du = 1 - e^{-\frac{\rho^2}{2t}}.$$  

14. Compute

$$\mathbb{E}_x \left( \int_{[0, \infty]} \chi_K(B_t) \, dt \right) = \int_{[0, \infty]} \mathbb{P}(B_t \in K) \, dt$$

$$= \int_{[0, \infty]} \left( \int_K \frac{1}{(2\pi t)^{n/2}} e^{-\frac{|x-y|^2}{2t}} d^n x \right) \, dt$$

$$\leq \int_{[0, \infty]} \left\| \frac{1}{(2\pi t)^{n/2}} e^{-\frac{|x-y|^2}{2t}} \right\|_\infty \mu(K) \, dt$$

$$= 0$$

and deduce that the expected total time spent in $K$ is $0$.

15. Note that $UU^T = I$, whence $|\det U| = 1$ and the probability measures are identical by change of variables. It follows that both are Brownian.

16. Let $W_t = \frac{1}{c} B_{c^2 t}$. We have $W_0 = B_0 = 0$ and that $W_t$ is absolutely continuous as a scaling of absolutely continuous $B_t$. Finally,

$$\mathbb{P}_0(W_t \in U) = \mathbb{P}_0(B_{c^2 t} \in cU)$$

$$= \int_{cU} p(c^2t, 0, y) \, dy$$

$$= \int_{cU} \frac{1}{c} p(t, 0, y/c) \, dy$$

$$= \int_U \frac{1}{c} p(t, 0, y')(cdy')$$

$$= \mathbb{P}_0(B_t \in U),$$

and so $W_t$ is also a Brownian motion.

17. Let $X_t(\cdot)$ be a continuous stochastic process.
(a) Recall that $\mathbb{E}(B_t) = 0$, $\mathbb{E}(B_t^2) = t$ and $\mathbb{E}(B_t^4) = 3t^2$. Then

$$
\mathbb{E} \left( \left( \sum_k (\Delta B_k - \Delta t_k)^2 \right) \right) = \mathbb{E} \left( \left( \sum_k (\Delta B_k^2 - \Delta t_k^2) \right) \right)
$$

$$
= \sum_k (\mathbb{E}(\Delta B_k^4) - 2\Delta t_k \mathbb{E}(\Delta B_k^2) + \Delta t_k^2)
$$

$$
= \sum_k (3\Delta t_k^2 - 2\Delta t_k^2 + \Delta t_k^2)
$$

$$
= 2 \sum_k \Delta t_k^2.
$$

So $\langle B, B \rangle_t^{(2)}(w) = t$.

(b) Note that the Brownian motion has positive quadratic variation $t$ on $[0, t]$. So

$$
\langle B, B \rangle_t^{(1)}(w) \geq \lim_{\|\Delta B_k\| \to 0^+, \|\Delta B_k\|} \frac{\langle B, B \rangle_t^{(2)}(w)}{\|\Delta B_k\|} = \infty.
$$
Chapter 3

Itô Integrals

1. Compute

\[
\int_0^t s \, dB_s = \lim_{n \to \infty} \sum_{j=0}^{\lfloor nt \rfloor - 1} \frac{j}{n} (B_{\frac{(j+1)t}{n}} - B_{\frac{jt}{n}}) = tB_t - \int_0^t B_s \, ds.
\]

2. Compute

\[
\int_0^t B_s^2 \, dB_s = \lim_{n \to \infty} \sum_{j=0}^{\lfloor nt \rfloor - 1} \frac{1}{n} B_{\frac{(j+1)t}{n}}^2 (B_{\frac{(j+1)t}{n}} - B_{\frac{jt}{n}}) = \frac{1}{3} B_t^3 - \int_0^t B_s \, ds.
\]

3. Let \( \{N_t\} \) be some filtration and let \( \{H_t^{(X)}\} \) be the filtration of process \( X_t \).

   (a) Compute

   \[
   \mathbb{E}(X_t \mid H_s^{(X)}) = \mathbb{E} \left( \mathbb{E}(X_t \mid N_s) \mid H_s^{(X)} \right) = \mathbb{E}(H_s \mid H_s^{(X)}) = H_s.
   \]
(b) Compute
\[ \mathbb{E}(X_t) = \mathbb{E}(\mathbb{E}(X_t \mid H_0^{(X)})) = \mathbb{E}(X_0). \]

(c) Let \( Y \sim \text{Bernoulli}(0.5) \) and fix \( X_0 = 2Y - 1 \). Then \( X_t = t \cdot \text{sgn}(X_0) \) satisfies \( \mathbb{E}(X_t) = \mathbb{E}(X_0) = 0 \), but \( \mathbb{E}(X_t \mid F_s) = t \cdot \text{sgn}(X_0) \neq s \cdot \text{sgn}(X_0) \).

4. Compute
\[
\mathbb{E}(B_{t} + 4t \mid F_s) = B_s + 4t \neq B_s + 4s \\
\mathbb{E}(B_t^2 \mid F_s) = \mathbb{E}((B_t - B_s)^2 + 2B_s(B_t - B_s) + B_s^2 \mid F_s) = B_s^2 + t - s \neq B_s^2 \\
\mathbb{E}\left(t^2B_t - 2\int_0^t uB_u \, du \mid F_s\right) = t^2B_s - 2\int_s^t uB_u \, du - 2\int_s^t uB_s \, du = s^2B_s - 2\int_0^s uB_u \, du \\
\mathbb{E}(B_t^{(1)}B_t^{(2)} \mid F_s) = \mathbb{E}(B_t^{(1)} \mid F_s)\mathbb{E}(B_t^{(2)} \mid F_s) = B_s^{(1)}B_s^{(2)},
\]
and deduce that only the last two are martingales.

5. Verify \( \mathbb{E}(|B_t^2 - t|) \leq \mathbb{E}(B_t^2) + t = 2t < \infty \) and compute
\[
\mathbb{E}(B_t^2 - t \mid F_s) = \mathbb{E}((B_t - B_s)^2 + 2B_s(B_t - B_s) + B_s^2 - t \mid F_s) = B_s^2 + t - s - t = B_s^2 - s.
\]
to deduce that \( X_t := B_t^2 - t \) is a martingale.

6. Verify \( \mathbb{E}(|B_t^3 - 3tB_t|) \leq \sqrt{\mathbb{E}(B_t^2)}(\sqrt{\mathbb{E}(B_t^4)} + 3t) = (3 + \sqrt{3})t^{3/2} < \infty \) and compute
\[
\mathbb{E}(B_t^3 - 3tB_t \mid F_s) = \mathbb{E}((B_t - B_s)^3 + 3B_s(B_t - B_s)^2 + 3B_s^2(B_t - B_s) + B_s^3 - 3tB_s \mid F_s) \\
= 3B_s(t - s) + B_s^3 - 3tB_s \\
= B_s^3 - 3sB_s
\]
to deduce that \( Y_t := B_t^3 - 3tB_t \) is a martingale.

7. In this question, the formula for Itô iterated integrals is derived.

(a) Note that \( \{0 \leq u_1 \cdots \leq u_n\} \) is Borel measurable and \( \chi_{0 \leq u_1 \cdots \leq u_n} \) is \( F_t \)-adapted. Finally
\[ \mathbb{E}\left(\int_0^T f(t_1, \ldots, t_n, \omega)^2 \, dt_1 \ldots dt_n\right) \leq T^n < \infty. \]

(b) For \( n \in \{1, 2, 3\} \)
\[
1! \int_0^t dB_u = B_t = t^{1/2}H_1\left(\frac{B_t}{\sqrt{t}}\right) \\
2! \int_0^t \int_0^v dB_u dB_v = 2 \int_0^t B_v dB_v = B_t^2 - t = tH_2\left(\frac{B_t}{\sqrt{t}}\right) \\
3! \int_0^t \int_0^w \int_0^v dB_u dB_v dB_w = 3 \int_0^t (B_w^2 - w) dB_w = B_t^3 - 3tB_t = t^{3/2}H_3\left(\frac{B_t}{\sqrt{t}}\right).
\]
(c) Deduce that \( dB_t^3 - 3tB_t \) = \( 3(B_t^2 - t) dB_t \) and so \( Y_t := B_t^3 - 3tB_t \) is a martingale.

8. There exists continuous martingale \( M_t \) iff there exists \( Y \in L^1 \) such that \( M_t = \mathbb{E}(Y \mid \mathcal{F}_t) \).

(a) Verify that \( \mathbb{E}(\mathbb{E}(Y \mid \mathcal{F}_t)) \leq \mathbb{E}(\mathbb{E}(Y \mid \mathcal{F}_t) = \mathbb{E}(Y) < \infty \) and

\[
\mathbb{E}(M_t \mid \mathcal{F}_s) = \mathbb{E}(\mathbb{E}(Y \mid \mathcal{F}_t) \mid \mathcal{F}_s) = \mathbb{E}(Y \mid \mathcal{F}_s) = M_s.
\]

(b) If \( M_t \) is a continuous martingale such that \( \sup_{t>0} \mathbb{E}(|X|^p) < \infty \) for \( p \in (1, \infty) \), then \( \exists M \) such that \( \|M_t - M\|_{L^1} \to 0 \) as \( t \to \infty \). So let \( Y = M \) and

\[
\lim_{s \to \infty} \int_{\Omega_s} |M_s - \mathbb{E}(M \mid \mathcal{F}_s)| \, dP = \lim_{s \to \infty} \int_{\Omega_s} |\mathbb{E}(M_s - M \mid \mathcal{F}_s)| \, dP \\
\leq \lim_{s \to \infty} \int_{\Omega_s} \mathbb{E}(|M_s - M| \mid \mathcal{F}_s) \, dP \\
= \lim_{s \to \infty} \int_{\Omega_s} |M_s - M| \, dP \\
= 0.
\]

9. Compute

\[
\int_0^T B_t \circ dB_t = \lim_{n \to \infty} \sum_{j=0}^{\lfloor nt \rfloor - 1} \frac{1}{2} (B_{\frac{j+1}{n} t} - B_{\frac{j}{n} t}) (B_{\frac{j+1}{n} t} - B_{\frac{j}{n} t}) \\
= \lim_{n \to \infty} \sum_{j=0}^{\lfloor nt \rfloor - 1} \frac{1}{2} (B_{\frac{j+1}{n}} - B_{\frac{j}{n}})^2 + \lim_{n \to \infty} \sum_{j=0}^{\lfloor nt \rfloor - 1} \frac{1}{2} (B_{\frac{j+1}{n}} - B_{\frac{j}{n}})^2 \\
= \frac{1}{2} B_t^2 - \frac{t}{2} + \frac{t}{2} \\
= \frac{1}{2} B_t^2.
\]

10. If \( f(t, \omega) \) varies smoothly in \( t \), then the Itô and Stratonovich integrals coincide. Compute

\[
\int_0^T f(t, \omega) \circ dB_t = \int_0^T f(t, \omega) dB_t + \frac{1}{2} \langle f(t, \omega), B_t \rangle^{(2)}
\]

and

\[
\mathbb{E}(\langle f(t, \omega), B_t \rangle^{(2)})^2 \leq \mathbb{E}(\langle B_t, B_t \rangle^{(2)}) \mathbb{E}(\langle f(t, \omega), f(t, \omega) \rangle^{(2)}) \\
\leq T \lim_{\|\Delta t_k\| \to 0^+} \sup_{|\Delta t_k|} \frac{T}{|\Delta t_k|} (K|\Delta t_k|^{1+\varepsilon}) \\
= KT^2 \lim_{\|\Delta t_k\| \to 0^+} \|\Delta t_k\|^\varepsilon \\
= 0.
\]
11. Define white noise \( W_t^{(N)} = \max\{-N, \min\{W_t, N\}\} \). Since \( W_t \) and \( W_s \) are independent and identically distributed, it follows that \( W_t^{(N)} \) and \( W_s^{(N)} \) are as well. If \( W_t \) is continuous, then since \( |W_t^{(N)}| \leq N \) and by bounded convergence
\[
\lim_{t \to s} \mathbb{E}(W_t^{(N)})^2 = \lim_{t \to s} \mathbb{E}(|W_t^{(N)} - W_s^{(N)}|^2) = 0.
\]
But then \( W_t \overset{a.s.}{=} 0 \), which is a contradiction.

12. Let \( \circ dB_t \) denote the Stratonovich differential.

   (i) Since \( \alpha X_t \circ dB_t = \frac{\alpha^2}{2} X_t dt + \alpha X_t dB_t \),
   \[
dX_t = (\gamma + \frac{\alpha^2}{2})X_t dt + \alpha X_t dB_t.
   \]
   Since \( (t^2 + \cos(X_t)) \circ dB_t = -\frac{\sin(X_t)}{2} (t^2 + \cos(X_t)) dt + (t^2 + \cos(X_t)) dB_t \),
   \[
dX_t = \frac{\sin(X_t)}{2} (\cos(X_t) - t^2) dt + (t^2 + \cos(X_t)) dB_t.
   \]

   (ii) Since \( \alpha X_t dB_t = \alpha X_t \circ dB_t - \frac{\alpha^2}{2} X_t dt \),
   \[
dX_t = (r - \frac{\alpha^2}{2}) X_t dt + \alpha X_t \circ dB_t.
   \]
   Since \( X_t^2 dB_t = X_t^2 \circ dB_t - X_t^3 dt \),
   \[
dX_t = (2e^{-X_t} - X_t^3) dt + X_t^2 \circ dB_t.
   \]

13. Let \( X_t \) be continuous in mean square. Calculate

   (a) \( \lim_{s \to t} \mathbb{E}[(B_t - B_s)^2] = \lim_{s \to t} \mathbb{E}[(B_t - s)^2] = \lim_{s \to t} (t - s) = 0 \)

   (b) \( \lim_{s \to t} \mathbb{E}[(f(B_t) - f(B_s))^2] \leq \lim_{s \to t} C^2 \mathbb{E}[(B_t - B_s)^2] = 0 \)

   (c) and finally by Itô isometry,

   \[
   \lim_{n \to \infty} \mathbb{E} \left[ \left( \int_{S}^{T} (X_s - \phi_n(s)) dB_s \right)^2 \right] = \lim_{n \to \infty} \mathbb{E} \left[ \int_{S}^{T} (X_s - \phi_n(s))^2 ds \right]
   \]
   \[
   = \lim_{n \to \infty} \mathbb{E} \left[ \sum_{j} \int_{t_j}^{t_{j+1}} (X_t - X_{t_j})^2 dt \right]
   \]
   \[
   \leq (T - S) \lim_{n \to \infty} \sup_{1 \leq j \leq n} \mathbb{E}[(X_t - X_{t_j})^2] = 0.
   \]
14. Show that \( h(\omega) \) is \( \mathcal{F}_t \) measurable if and only if it is the pointwise limit of a sum-product of bounded continuous functions \( g(B_{t_j}) \).

(a) Assume that \( h \) is bounded since \( \{ h_n(\omega) := h(\omega) \mathbb{1}_{\{ |h(\omega)| < n \}} \} \) converges pointwise to \( h \).

(b) Let \( \mathcal{H}_n \) be the \( \sigma \)-algebra generated by \( B(t_j) \) for \( t_j = \frac{j}{2^n} \leq t \). Then \( \mathcal{F}_t = \sigma (\cup_n \mathcal{H}_n) \) and so by Corollary (C.9), \( h = \mathbb{E}[h|\mathcal{F}_n] = \lim_{n \to \infty} \mathbb{E}[h|\mathcal{H}_n] \).

(c) By Doob-Dynkin, \( \mathbb{E}[h|\mathcal{H}_n](\omega) = g(B_{t_1}, \ldots , B(t_{2^n t})) \). Since \( C(\mathbb{R}^k) \) is dense in \( L^1(\mathbb{R}^k) \) and by Stone-Weierstrass \( P(\mathbb{R}^k) \) is dense in \( C(\mathbb{R}^k) \), a limiting sequence must exist.

15. Suppose \( C + \int_S^T f(t,\omega) \, dB_t(\omega) = D + \int_S^T g(t,\omega) \, dB_t(\omega) \). Then we have that

\[
C - D = \mathbb{E}[C - D] = \mathbb{E} \left[ \int_S^T g(t,\omega) \, dB_t(\omega) - \int_S^T f(t,\omega) \, dB_t(\omega) \right] = 0 \implies C = D,
\]

and by Itô isometry,

\[
0 = \mathbb{E} \left[ \left( \int_S^T g(t,\omega) \, dB_t(\omega) - \int_S^T f(t,\omega) \, dB_t(\omega) \right)^2 \right] = \int_S^T \mathbb{E}[(g(t,\omega) - f(t,\omega))^2] \, ds,
\]

whence \( g(t,\omega) = f(t,\omega) \) almost surely for \( (t,\omega) \in [S, T] \times \Omega \).

16. By Jensen’s inequality, \( \mathbb{E} [\mathbb{E}[X|\mathcal{H}]^2] \leq \mathbb{E} [\mathbb{E}[X^2|\mathcal{H}]] = \mathbb{E}[X^2] \).

17. Let \( \mathcal{G} \) be a finite \( \sigma \)-algebra with partition \( \Omega = \bigsqcup_{i=1}^n G_i \).

(a) Note that \( \mathbb{E}[X|\mathcal{G}](\omega) = \sum_{i=1}^n c_i \mathbb{1}_{G_i}(\omega) = c_i \) on \( G_i \).

(b) Show that

\[
\int_{G_i} \left( \frac{\int_{G_i} X \, d\mathbb{P}}{\mathbb{P}(G_i)} \right) d\mathbb{P} = \int_{G_i} X \, d\mathbb{P} = \int_{G_i} d\mathbb{P}, \quad \forall i \in \{1, \ldots , n\}.
\]

(c) By part (b), \( c_i = \frac{\int_{G_i} X \, d\mathbb{P}}{\mathbb{P}(G_i)} \). Show for \( \omega \in G_i \) that

\[
\mathbb{E}[X|\mathcal{G}](\omega) = \sum_{i=1}^n \frac{\int_{G_i} X \, d\mathbb{P}}{\mathbb{P}(G_i)} \mathbb{1}_{G_i}(\omega)
= \frac{\int_{G_i} X \, d\mathbb{P}}{\mathbb{P}(G_i)}
= \frac{\sum_{k=1}^m a_k \mathbb{P}(X = a_k, \omega \in G_i)}{\mathbb{P}(G_i)}
= \sum_{k=1}^m a_k \mathbb{P}(X = a_k|G_i).
\]
Chapter 4

The Itô Formula

1. Compute

(a) \( dX_t = d(B_t^2) = 2B_t dB_t + d[B, B]_t = 2B_t dB_t + dt \)
(b) \( dX_t = d(2 + t + e^{B_t}) = (1 + \frac{1}{2}e^{B_t}) dt + e^{B_t} dB_t \)
(c) \( dX_t = d \left( (B_t^{(1)})^2 + (B_t^{(2)})^2 \right) = 2B_t^{(1)} dB_t^{(1)} + 2B_t^{(2)} dB_t^{(2)} + 2 dt \)
(d) \( dX_t = d((t_0 + t, B_t)) = (dt, dB_t) \)
(e) and finally

\[
dX_t = d((B_t^{(1)} + B_t^{(2)} + B_t^{(3)}, (B_t^{(2)})^2 - B_t^{(1)} B_t^{(3)})) = (dB_t^{(1)} + dB_t^{(2)} + dB_t^{(3)}, 2B_t^{(2)} dB_t^{(2)} + dt - B_t^{(3)} dB_t^{(1)} - B_t^{(1)} dB_t^{(3)}).\]

2. Using Itô’s Lemma, differentiate

\[
d \left( \frac{1}{3}B_t^3 - \int_0^t B_s ds \right) = B_t^2 dB_s + B_t d[B, B]_t - B_t dt = B_t^2 dB_t
\]

and deduce that

\[
\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds.
\]

3. Let \( X_t \) and \( Y_t \) be Itô processes. Then, letting \( f(t, x, y) = xy \) and by Itô’s formula

\[
d(X_t Y_t) = f_t(t, X_t, Y_t) dt + f_x(t, X_t, Y_t) dX_t + f_y(t, X_t, Y_t) dY_t
\]

\[
+ \frac{1}{2} f_{xx}(t, X_t, Y_t) d[X, X]_t + f_{xy}(t, X_t, Y_t) d[X, Y]_t + \frac{1}{2} f_{yy}(t, X_t, Y_t) d[Y, Y]_t
\]

\[
= Y_t dX_t + X_t dY_t + d[X, Y]_t
\]
and deduce the integration of parts formula

\[
\int_0^t X_s \, dY_s = \int_0^t (d(X_s Y_s) - Y_s \, dX_s - d[X,Y]_s)
= X_t Y_t - X_0 Y_0 - \int_0^t Y_s \, dX_s - \int_0^t d[X,Y]_s.
\]

4. Let \( Z_t = \exp \left( \int_0^t \langle \theta(s, \omega), dB_s \rangle - \frac{1}{2} \theta(s, \omega)|^2 \, ds \right) \).

(a) Then, letting \( Z_t = e^{Y_t} \) and by Itô’s formula,

\[
dZ_t = e^{Y_t} \, dY_t + \frac{1}{2} e^{Y_t} \, d[Y,Y]_t
= Z_t \left( \langle \theta(t, \omega), dB_t \rangle - \frac{1}{2} \theta(t, \omega)^2 \, dt + \frac{1}{2} \sum_{i,j=1}^n \left[ \theta_i(s, \omega) \, dB^{(i)}_s, \theta_j(s, \omega) \, dB^{(j)}_s \right]_s \right)
= Z_t \langle \theta(t, \omega), dB_t \rangle.
\]

(b) It suffices to check that

\[
[\mathbb{E}(|Z_t|)]^2 = \left[ \mathbb{E} \left( \left| \int_0^t dZ_s \right| \right) \right]^2
= \left[ \mathbb{E} \left( \left| \int_0^t Z_s \langle \theta(s, \omega), dB_s \rangle \right| \right) \right]^2
\leq \mathbb{E} \left( \int_0^t \sum_{i=1}^n |Z_s \theta_i(s, \omega)| \, dB^{(i)}_s \right)^2
= \mathbb{E} \left( \sum_{i,j=1}^n \int_0^t |Z_s \theta_i(s, \omega)| \, |Z_s \theta_j(s, \omega)| \, d[B^{(i)}, B^{(j)}]_s \right)
= \sum_{i=1}^n \mathbb{E} \left( \int_0^t |Z_s \theta_i(s, \omega)|^2 \, ds \right)
< \infty.
\]

5. Let \( \beta_k(t) = \mathbb{E}(B^k_t) \). Then, by Itô’s lemma,

\[
\Delta B^k_t = k B^{k-1}_t \, dB_t + \frac{1}{2} k(k - 1) B^{k-2}_t \, dt
\]

and so

\[
\Delta \beta_k(t) = \mathbb{E}(B^k_t) = \mathbb{E} \left( \int_0^t dB^k_s \right) = \int_0^t \mathbb{E} \left( \frac{1}{2} k(k - 1) B^{k-2}_s \right) \, ds = \frac{1}{2} k(k - 1) \int_0^t \beta_{k-2}(s) \, ds.
\]

Deduce that \( \beta_4(t) = 6 \int_0^t \beta_2(s) \, ds = 6 \cdot \frac{t^2}{2} = 3t^2 \) and \( \beta_6(t) = 15 \int_0^t 3s^2 \, ds = 15t^3. \)
6. Define geometric Brownian motions $X_t = e^{ct+\alpha B_t}$ and $Y_t = e^{ct+\sum_{j=1}^{\infty} \alpha_j B_t^{(j)}}$.

(a) Calculate
\[
dX_t = ce^{ct+\alpha B_t} \, dt + \alpha e^{ct+\alpha B_t} \, dB_t + \frac{1}{2} \alpha^2 e^{ct+\alpha B_t} \, dB_t^2,
\[
= X_t \left( (c + \frac{\alpha^2}{2}) \, dt + \alpha \, dB_t \right).
\]

(b) Calculate
\[
dY_t = Y_t \left( c \, dt + \sum_{j=1}^{\infty} \alpha_j dB_t^{(j)} + \frac{1}{2} \sum_{i,j=1}^{\infty} \alpha_i \alpha_j \, dB_t^{(i)} dB_t^{(j)} \right),
\[
= Y_t \left( (c + \frac{1}{2} \sum_{j=1}^{\infty} \alpha_j^2) \, dt + \sum_{j=1}^{\infty} \alpha_j dB_t^{(j)} \right).
\]

7. Let $X_t$ solve $dX_t = v(t,\omega) \, dB_t$.

(a) Note that $B_t$ is a martingale while $B_t^2$ is not.

(b) Define $M_t = X_t^2 - \int_0^t v(s,\omega)^2 \, ds$. Then
\[
dM_t = 2X_t \, dX_t + [dX, \, dX]_t - v(t,\omega)^2 \, dt
\[
= 2X_t v(t,\omega) \, dB_t + (v(t,\omega)^2 - v(t,\omega)^2) \, dt
\[
= 2X_t v(t,\omega) \, dB_t.
\]

Moreover,
\[
\mathbb{E}(|M_t|) \leq \mathbb{E}(X_t^2) + \mathbb{E} \left( \int_0^t v(s,\omega)^2 \, ds \right)
\]
\[
= \mathbb{E} \left( \int_0^t v(s,\omega) \, dB_s \right)^2 + \mathbb{E} \left( \int_0^t v(s,\omega)^2 \, ds \right)
\]
\[
= 2 \mathbb{E} \left( \int_0^t v(s,\omega)^2 \, ds \right)
\]
\[
< \infty.
\]

8. Let $f(x^{(1)}, \ldots x^{(n)})$ be a function of class $C^2$.

(a) By Itô’s lemma,
\[
d(f(B_t)) = \sum_{i=1}^{n} \partial_i f(B_t) \, dB_t^{(i)} + \frac{1}{2} \sum_{i,j=1}^{n} \partial_{ij}^2 f(B_t) \, dB_t^{(i)} dB_t^{(j)}
\]
\[
= \langle \nabla f(B_t), dB_t \rangle + \frac{1}{2} \Delta f(B_t) \, dt
\]
and so
\[ f(B_t) - f(B_0) = \int_0^t d(f(B_s)) = \int_0^t \langle \nabla f(B_s), dB_s \rangle + \frac{1}{2} \int_0^t \Delta f(B_s) \, ds. \]

(b) Assume that \( g \) is of class \( C^1 \) everywhere, as well as \( C^2 \) and uniformly bounded outside of finitely many points with \( |g''(z)| \leq M \) for \( z \notin \{z_1, \ldots, z_k\} \). Then the set of functions \( \{f\} \) of class \( C^2 \) uniformly bounded with \( |f''(z)| \leq M \) are \( C^k \)-dense. So we can extract a sequence \( \{f_k\} \) such that \( f_k \Rightarrow g, f'_k \Rightarrow g' \) as well as \( f''_k \to g'' \) and \( |f''_k| \leq M \) on \( \mathbb{R} \setminus \{z_1, \ldots, z_k\} \). So
\[
\lim_{k \to \infty} \left| f_k(B_t) - f(B_t) + f_k(0) - f(0) + \int_0^t (f'_k - g') \, dB_s + \frac{1}{2} \int_0^t (f''_k - g'') \, ds \right| \\
\leq \lim_{k \to \infty} \left| (f_k - g)(B_t) + (f_k - g)(0) \right| + \int_0^t \left| f'_k - g' \right| \, dB_s + \frac{1}{2} \int_0^t \left| f''_k - g'' \right| \, ds \\
= 0,
\]

where the last term vanishes by bounded convergence.

9. Clearly
\[
\int_0^t v \frac{\partial g_n}{\partial x}(s, X_s) \chi_{s \leq \tau_n} \, dB_s = \int_0^{t \wedge \tau_n} v \frac{\partial g}{\partial x}(s, X_s) \, dB_s
\]
and the result follows by Itô’s lemma where \( dB_t = ud\tau + dB_t \). Since \( \mathbb{E}(|X_t|) < \infty \), it follows that \( \lim_{n \to \infty} \mathbb{P}(\tau_n > t) = \lim_{n \to \infty} \mathbb{P}(X_t < n) = 1 \) and so the identity holds almost surely.

10. (Tanaka) In this problem, Tanaka’s formula for Brownian motion is derived.

(a) Substitute \( u \equiv 0 \) and \( v \equiv 1 \) here. Then as \( g''_\varepsilon(x) = \frac{1}{2\varepsilon} \chi_{|x|<\varepsilon}(x) \)
\[
\frac{1}{2} \int_0^t \frac{d^2g_\varepsilon}{dx^2}(B_s) \, ds = \frac{1}{2\varepsilon} \int_0^t \chi_{|B_s|<\varepsilon} \, ds = \frac{1}{2\varepsilon} \left| \{s \in [0,t] \mid |B_s| < \varepsilon \} \right|.
\]
(b) Differentiate to get
\[
\int_0^t g'_\varepsilon(B_s) \chi_{|B_s|<\varepsilon} \, dB_s = \int_0^t \frac{B_s}{\varepsilon} \chi_{|B_s|<\varepsilon} \, dB_s,
\]
and apply Itô isometry to get
\[
\lim_{\varepsilon \to 0^+} \mathbb{E} \left( \int_0^t \frac{B_s}{\varepsilon} \chi_{|B_s|<\varepsilon} \, dB_s \right)^2 = \lim_{\varepsilon \to 0^+} \mathbb{E} \left( \int_0^t \frac{B_s^2}{\varepsilon^2} \chi_{|B_s|<\varepsilon} \, ds \right) \leq \lim_{\varepsilon \to 0^+} \int_0^t \mathbb{P}(|B_s| < \varepsilon) \, ds = 0.
\]
11. Let \( X_t = e^{t/2} \cos(B_t), \ Y_t = e^{t/2} \sin(B_t) \) and \( Z_t = (B_t + t)e^{-B_t-t/2} \). Compute
(a) \( dX_t = \frac{1}{2}e^{t/2} \cos(B_t) \, dt - e^{t/2} \sin(B_t) \, dB_t + \frac{1}{2}(-e^{t/2} \cos(B_t)) \, d[B, B]_t = -e^{t/2} \sin(B_t) \, dB_t \)
(b) \( dY_t = \frac{1}{2}e^{t/2} \sin(B_t) \, dt + e^{t/2} \cos(B_t) \, dB_t + \frac{1}{2}(-e^{t/2} \sin(B_t)) \, d[B, B]_t = e^{t/2} \cos(B_t) \, dB_t \)
(c) As \( \varepsilon \to 0 \) for \( g(x) = x \),
\[
|B_t| = |B_0| + \lim_{\varepsilon \to 0^+} \int_0^t \text{sgn}(B_s) \chi_{|B_s| \geq \varepsilon} \, ds + \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \{ s \in [0, t] \mid |B_s| < \varepsilon \}.
\]
\[
= |B_0| + \int_0^t \text{sgn}(B_s) \, ds + L_t.
\]
12. The given condition implies \( \mathbb{E}(|X_t|) < \infty \). So \( X_t \) is a martingale if and only if \( \mathbb{E}(X_t \mid \mathcal{F}_s) = X_s \). Then
\[
\mathbb{E}(\int_s^t u(r, \omega) \, dr \mid \mathcal{F}_s) = \mathbb{E}(X_t - X_s \mid \mathcal{F}_s) = 0.
\]
Moreover by dominated convergence
\[
\mathbb{E}(u(t, \omega) \, d \mid \mathcal{F}_s) = \mathbb{E}(\frac{d}{ds} \int_s^t u(r, \omega) \, dr \mid \mathcal{F}_s) = 0.
\]
Then
\[
u(t, \omega) = \mathbb{E}(u(t, \omega) \mid \mathcal{F}_t) = \lim_{s \to t^-} \mathbb{E}(u(t, \omega) \mid \mathcal{F}_s) = 0.
\]
13. Let \( dX_t = u(t, \omega) \, dt + dB_t \) where \( u(t, \omega) \in \mathcal{V}([0, T]) \). Then \( Y_t = X_tM_t \) is a martingale, where
\[
M_t = \exp \left( -\int_0^t u(r, \omega) \, dB_r - \frac{1}{2} \int_0^t u^2(r, \omega) \, dr \right)
\]
since \( \mathbb{E}(|M_t|) < \infty \) (see question 4b), \( \mathbb{E}(|X_t|) \leq \sqrt{t} \left( \sqrt{\int_0^t u^2(r, \omega) \, dr} + 1 \right) < \infty \) and
\[
d(X_tM_t) = M_t \, dX_t + X_t \, dM_t + d[X, M]_t
\]
\[
= M_t(u(t, \omega) \, dt + dB_t) + M_tX_t(-u(t, \omega) \, dB_t - \frac{1}{2}u^2(t, \omega) \, dt)
\]
\[
- M_t u(t, \omega) \, dt + \frac{1}{2} M_tX_t u^2(t, \omega) \, dt
\]
\[
= M_t(1 - u(t, \omega)X_t) \, dB_t.
\]
14. In this problem, the martingale representation of stochastic processes is explicitly shown.

(a) Compute \( dF_t = dB_t, \mathbb{E}(F_T) = 0 \) and
\[
dF_t - d\mathbb{E}(F_t) = 1 dB_t \implies f(t, \omega) = 1.
\]

(b) Compute \( dF_t = B_t \, dt, \mathbb{E}(F_T) = 0 \) and
\[
dF_t - d\mathbb{E}(F_t) = B_t \, dt = dB_t = (T-t) \, dB_t \implies f(t, \omega) = T-t.
\]

(c) Compute \( dF_t = 2 B_t \, dB_t + \, dt, \mathbb{E}(F_T) = T \) and
\[
dF_t - d\mathbb{E}(F_t) = 2 B_t \, dB_t + dB_t = (2 B_t + 1 \cdot T) \, dB_t = (2 B_t + 3(T-t)) \, dB_t \implies f(t, \omega) = 2 B_t + 3(T-t).
\]

(d) Compute \( dF_t = 3 B_t^2 \, dB_t + 3 B_t \, dt, \mathbb{E}(F_T) = 0 \) and
\[
dF_t - d\mathbb{E}(F_t) = 3 B_t^2 \, dB_t + 3 B_t \, dt = 3 B_t^2 + 3(T-t) \, dB_t \implies f(t, \omega) = 3 B_t^2 + 3(T-t).
\]

(e) Recall that \( e^{B_{t-t/2}} \) is a martingale and compute
\[
d(e^{B_{t-t/2}}) = e^{B_{t-t/2}} dB_t.
\]
Deduce that
\[
e^{B_T} = e^{T/2} \left( 1 + \int_0^T e^{B_{t-t/2}} dB_t \right) \implies f(t, \omega) = e^{B_t + (T-t)/2}.
\]

(f) Find martingale \( e^{t/2} \sin(B_t) \) and compute
\[
d(e^{t/2} \sin(B_t)) = e^{t/2} \cos(B_t) dB_t
\]
Deduce that
\[
\sin(B_T) = e^{-T/2} \int_0^T e^{t/2} \cos(B_t) dB_t \implies f(t, \omega) = e^{-(T-t)/2} \cos(B_t).
\]

15. Define \( X_t = (x^{1/3} + \frac{1}{3} B_t)^3 \). Then
\[
dX_t = 3X_t^{2/3} \, d(x^{1/3} + \frac{1}{3} B_t) + 3X_t^{1/3} \, d \left[ x^{1/3} + \frac{1}{3} B_t, x^{1/3} + \frac{1}{3} B_t \right]
\]
\[
= X_t^{2/3} dB_t + \frac{1}{3} X_t^{1/3} \, dt.
\]
Chapter 5

Stochastic Differential Equations

1. Compute

(a) \( dX_t = d(e^{B_t}) = e^{B_t} dB_t + \frac{1}{2} e^{B_t} d[B,B]_t = \frac{1}{2} X_t dt + X_t dB_t \)
(b) \( dX_t = d\left( \frac{B_t}{1+t} \right) = \frac{1}{1+t} dB_t - \frac{B_t}{(1+t)^2} dt = \frac{1}{1+t} dB_t - \frac{1}{1+t} X_t dt \)
(c) \( dX_t = d(\sin(B_t)) = \cos(B_t) dB_t - \frac{1}{2} \sin(B_t) dt = \cos(B_t) dB_t - \frac{1}{2} X_t dt \)
(d) \( dX^{(1)}_t = dt \) and
   \( dX^{(2)}_t = d(e^t B_t) = e^t dB_t + e^t B_t dt = e^t dB_t + X^{(2)}_t dt \).
(e) and finally differentials

\[
d(cosh(B_t)) = sinh(B_t) dB_t + \frac{1}{2} cosh(B_t) dt
\]

and

\[
d(sinh(B_t)) = cosh(B_t) dB_t + \frac{1}{2} sinh(B_t) dt
\]

to deduce that

\[
\begin{pmatrix} dX^{(1)}_t \\ dX^{(2)}_t \end{pmatrix} = \frac{1}{2} \begin{pmatrix} X^{(1)}_t \\ X^{(2)}_t \end{pmatrix} dt + \begin{pmatrix} X^{(2)}_t \\ X^{(1)}_t \end{pmatrix} dB_t.
\]

2. Let \( X^{(1)}_t = a \cos(B_t) \) and \( X^{(2)}_t = b \sin(B_t) \). Then

\[
dX^{(1)}_t = -a \sin(B_t) dB_t - \frac{a}{2} \cos(B_t) dt = -\frac{1}{2} X^{(1)}_t dt - \frac{a}{b} X^{(2)}_t dB_t
\]

and

\[
dX^{(2)}_t = b \cos(B_t) dB_t - \frac{b}{2} \sin(B_t) dt = -\frac{1}{2} X^{(2)}_t dt + \frac{b}{a} X^{(1)}_t dB_t.
\]
3. The solution is given by

\[ X_t = X_0 \exp \left( (r - \frac{1}{2} \sum_{k=1}^{n} \alpha_k^2) t + \sum_{k=1}^{n} \alpha_k dB_k \right). \]

4. In this problem, solutions to stochastic differential equations are found.

(a) The solution to \( dX^{(1)}_t = dt + dB^{(1)}_t \) is \( X^{(1)}_t = X^{(1)}_0 + t + B^{(1)}_t \) and

\[ dX^{(2)}_t = X^{(1)}_t dB^{(2)}_t = (X^{(1)}_0 + t + B^{(1)}_t) dB^{(2)}_t \]

is

\[ X^{(2)}_t = X^{(2)}_0 + X^{(1)}_0 B^{(2)}_t + \int_0^t (s + B^{(1)}_s) dB^{(2)}_s. \]

(b) Using integrating factors, solve \( dX_t = X_t dt + dB_t \) for

\[ e^{-t} X_t - X_0 = \int_0^t e^{-s} dB_s \]

and deduce that the solution \( X_t \) is

\[ X_t = e^t X_0 + \int_0^t e^{t-s} dB_s. \]

(c) Using integrating factors, solve \( dX_t = -X_t dt + e^{-t} dB_t \) for

\[ e^t X_t - X_0 = \int_0^t dB_s \]

and deduce that the solution \( X_t \) is

\[ X_t = e^{-t} (X_0 + B_t). \]

5. The Langevin equation is given by

\[ dX_t - \mu X_t dt = \sigma dB_t. \]

(a) Using integrating factors, solve for

\[ e^{-\mu t} X_t - X_0 = \int_0^t e^{-\mu s} \sigma dB_s \]

and deduce that the solution \( X_t \) is

\[ X_t = e^{\mu t} X_0 + \sigma \int_0^t e^{\mu (t-s)} dB_s. \]
(b) The expected value of $X_t$ is

$$\mathbb{E}(X_t) = e^{\mu t} X_0$$

and, by Itô isometry, the variance of $X_t$ is

$$\text{Var}(X_t) = \mathbb{E} \left( \sigma^2 \int_0^t e^{\mu (t-s)} \, dB_s \right)^2 = \mathbb{E} \left( \sigma^2 \int_0^t e^{2\mu (t-s)} \, ds \right) = \frac{\sigma^2}{2 \mu} (e^{2\mu t} - 1).$$

6. Suppose $Y_t$ is given by

$$dY_t = r \, dt + \alpha Y_t \, dB_t.$$ 

Using integrating factors, solve for

$$d(e^{-\alpha B_t} Y_t) = e^{-\alpha B_t} Y_t \left( r - \frac{\alpha^2}{2} \right) \, dt$$

and

$$e^{-\alpha B_t + \frac{\alpha^2}{2} t} Y_t - Y_0 = \int_0^t r e^{-\alpha B_s + \frac{\alpha^2}{2} s} \, ds.$$ 

Deduce that

$$Y_t = e^{\alpha B_t - \frac{\alpha^2}{2} t} Y_0 + r \int_0^t e^{\alpha (B_t - B_s) - \frac{\alpha^2}{2} (t-s)} \, ds.$$ 

7. The Ornstein-Uhlenbeck process is given by

$$dX_t = (m - X_t) \, dt + \sigma \, dB_t.$$ 

(a) Using integrating factors, solve for

$$e^{t} X_t - X_0 = \int_0^t e^s m \, ds + \int_0^t e^s \sigma \, dB_s$$

and deduce that the solution $X_t$ is

$$X_t = e^{-t} X_0 + m(1 - e^{-t}) + \sigma \int_0^t e^{s-t} \, dB_s.$$ 

(b) The expected value of $X_t$ is

$$\mathbb{E}(X_t) = m + e^{-t}(X_0 - m)$$

and the variance of $X_t$ is

$$\text{Var}(X_t) = \mathbb{E} \left( \sigma^2 \int_0^t e^{s-t} \, dB_s \right)^2 = \mathbb{E} \left( \sigma^2 \int_0^t e^{2s-2t} \, ds \right) = \frac{\sigma^2}{2} (1 - e^{-2t}).$$
8. Consider the stochastic differential equation
\[
\begin{pmatrix}
    dX^{(1)}_t \\
    dX^{(2)}_t
\end{pmatrix} =
\begin{pmatrix}
    0 & 1 \\
    -1 & 0
\end{pmatrix}
\begin{pmatrix}
    X^{(1)}_t \\
    X^{(2)}_t
\end{pmatrix} dt +
\begin{pmatrix}
    \alpha dB^{(1)}_t \\
    \beta dB^{(2)}_t
\end{pmatrix}.
\]

By d’Alembert’s formula, it has a solution of the form
\[
X_t = e^{At} X_0 + \int_0^t e^{A(t-s)} g(s) \, ds,
\]
where
\[
e^{At} = \begin{pmatrix} 1 & 1 \\ i & -1 \end{pmatrix} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.
\]

Conclude that the solutions are
\[
X^{(1)}_t = X^{(1)}_0 \cos(t) + X^{(2)}_0 \sin(t) + \alpha \int_0^t \cos(t-s) dB^{(1)}_s + \beta \int_0^t \sin(t-s) dB^{(2)}_s
\]
and
\[
X^{(2)}_t = -X^{(1)}_0 \sin(t) + X^{(2)}_0 \cos(t) - \alpha \int_0^t \sin(t-s) dB^{(1)}_s + \beta \int_0^t \cos(t-s) dB^{(2)}_s.
\]

9. Let \( dX_t = \ln(1 + X_t^2) \, dt + \chi_{\{X_t > 0\}} X_t \, dB_t \). It suffices to check that
\[
|b(t, x) + |\sigma(t, x)| = \ln(1 + x^2) + \chi_{\{x > 0\}} |x| \leq \frac{2}{e} (|x| + 1) + |x| \leq 2(|x| + 1),
\]
\[
E(|X_0|^2) = \alpha^2 < \infty, \quad \text{and}
\]
\[
|b(t, x) - b(t, y) + |\sigma(t, x) - \sigma(t, y)| \leq |\ln(x^2) - \ln(y^2)| + |x - y| \leq 3|x - y|.
\]

Hence, by Theorem 5.2.1, there is a unique strong solution to the stochastic differential equation.

10. Calculate
\[
E(X_t^2) = E \left( Z + \int_0^t b(s, X_s) \, ds + \int_0^t \sigma(s, X_s) \, dB_s \right)^2
\]
\[
\leq 3 \left( E(Z^2) + E \left( \int_0^t b(s, X_s) \, ds \right)^2 + E \left( \int_0^t \sigma(s, X_s) \, dB_s \right)^2 \right)
\]
\[
\leq 3 \left( E(Z^2) + T E \left( \int_0^t b(s, X_s)^2 \, ds \right) + E \left( \int_0^t \sigma(s, X_s)^2 \, ds \right) \right)
\]
\[
\leq 3E(Z^2) + 6C^2 \left( T + \int_0^t E(|X_s|^2) \, ds \right) (T + 1)
\]
\[
= (3E(Z^2) + 6C^2T(T + 1) + 6C^2(T + 1) \int_0^t E(|X_s|^2) \, ds.
\]

and apply Gronwall to derive the result.
11. Consider the stochastic process

\[ Y_t = a(1 - t) + bt + (1 - t) \int_0^t \frac{dB_s}{1 - s}. \]

Then \( Y_0 = a \) and, for \( t \in [0, 1) \), \( Y_t \) solves

\[ dY_t = (b - a) dt - \int_0^t \frac{dB_s}{1 - s} dt + (1 - t) \int_0^t \frac{dB_s}{1 - s} \]

\[ = \frac{1}{1 - t} \left( (b - a)(1 - t) - (1 - t) \int_0^t \frac{dB_s}{1 - s} \right) dt + dB_t \]

\[ = \frac{1}{1 - t} \left( b - a(1 - t) - bt - (1 - t) \int_0^t \frac{dB_s}{1 - s} \right) dt + dB_t \]

\[ = \frac{b - Y_t}{1 - t} dt + dB_t. \]

Finally by Itô isometry \( \mathbb{E} \left[ (1 - t)^2 \int_0^t \frac{dB_s}{1 - s} \right]^2 = (1 - t)^2 \int_0^t \frac{1}{(1 - s)^2} ds = (1 - t)t \to 0 \) as \( t \to 1^- \) and so limit \( \lim_{t \to 1^-} Y_t \to b. \)

12. Let \( y''(t) + (1 + \varepsilon W_t)y(t) = 0 \) where \( W_t = \frac{dB_t}{dt} \) is 1-dimensional white noise.

(a) Rewrite

\[ \begin{pmatrix} dy_t \\ dy_t \\ \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_t \\ \dot{y}_t \end{pmatrix} dt + \begin{pmatrix} 0 & 0 \\ -\varepsilon & 0 \end{pmatrix} \begin{pmatrix} y_t \\ \dot{y}_t \end{pmatrix} dB_t. \]

(b) Check that, if \( y(t) = y(0) + y'(0)t + \int_0^t (r - t)y(r) \, dr + \int_0^t \varepsilon(r - t)y(r) \, dB_r, \) then

\[ y'(t) = y'(0) - \int_0^t y(r) \, dr - \int_0^t \varepsilon y(r) \, dB_r = y'(0) - \int_0^t y(r)(1 + \varepsilon W_r) \, dr \]

and \( y''(t) = -(1 + \varepsilon W_r) \, dr. \)

13. Let \( x''_t + a_0 x'_t + w^2 x_t = (T_0 - \alpha_0 x'_t) \eta W_t \) where \( W_t \) is 1-dimensional white noise. Then

\[ \begin{pmatrix} dx_t \\ d\dot{x}_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -w^2 & -a_0 \end{pmatrix} \begin{pmatrix} x_t \\ \dot{x}_t \end{pmatrix} dt + \begin{pmatrix} 0 & 0 \\ 0 & -\alpha_0 \eta \end{pmatrix} \begin{pmatrix} x_t \\ \dot{x}_t \end{pmatrix} dB_t + \begin{pmatrix} 0 \\ T_0 \eta \end{pmatrix} dB_t \]

and by d’Alembert’s formula the solution is

\[ X_t = e^{At} X_0 + \int_0^t e^{A(t-s)} KX_s dB_s + \int_0^t e^{A(t-s)} M \, dB_s. \]
The eigenvalues of $A$ satisfy $\lambda^2 + a_0 \lambda + w^2 = 0$ and are $\lambda_{\pm} = -\frac{a_0}{2} \pm \sqrt{w^2 - \frac{a_0^2}{4}}i =: -\lambda \pm \xi i$. Then take the exponential of matrix $A$

$$e^{At} = \begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix} \begin{pmatrix} e^{\lambda_+ t} & 0 \\ 0 & e^{\lambda_- t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix}^{-1}$$

$$= \frac{1}{\lambda_- - \lambda_+} \begin{pmatrix} \lambda_- e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t} & e^{\lambda_+ t} - e^{\lambda_- t} \\ -\lambda_- e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t} & \lambda_- e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t} \end{pmatrix}$$

$$= \frac{1}{2\xi i} \begin{pmatrix} e^{-\lambda t}(-\lambda \cdot 2i \sin(\xi t) - \xi i \cdot 2 \cos(\xi t)) & e^{-\lambda t}(-\lambda \cdot 2i \sin(\xi t) - \xi i \cdot 2 \cos(\xi t) + 2\lambda \cdot 2i \sin(\xi t)) \\ -w^2 e^{-\lambda t}(-2 \sin(\xi t)) & e^{-\lambda t}(-\lambda \cdot 2i \sin(\xi t) - \xi i \cdot 2 \cos(\xi t) + 2\lambda \cdot 2i \sin(\xi t)) \end{pmatrix}$$

$$= \frac{e^{-\lambda t}}{\xi} \begin{pmatrix} \lambda \sin(\xi t) + \xi \cos(\xi t) & \sin(\xi t) \\ -w^2 \sin(\xi t) & \lambda \sin(\xi t) + \xi \cos(\xi t) - 2\lambda \sin(\xi t) \end{pmatrix}$$

$$= \frac{e^{-\lambda t}}{\xi} \left( (\lambda \sin(\xi t) + \xi \cos(\xi t)) I + A \sin(\xi t) \right).$$

Next, let $y_s = \dot{x}_s$, $g_t = e^{-\lambda t \sin(\xi t)}$ and $h_t = e^{-\lambda t \xi \sin(\xi t)}$, compute

$$e^{A(t-s)} K X_s = -\frac{\alpha_0 \eta e^{-\lambda(t-s)}}{\xi} \begin{pmatrix} 0 & \sin(\xi(t-s)) \\ 0 & \xi \cos(\xi(t-s)) - \lambda \sin(\xi(t-s)) \end{pmatrix} \begin{pmatrix} x_s \\ \dot{x}_s \end{pmatrix} = \begin{pmatrix} -\alpha_0 \eta y_s g_{t-s} \\ -\alpha_0 \eta y_s h_{t-s} \end{pmatrix}$$

and

$$e^{A(t-s)} M = \frac{T_0 \eta e^{-\lambda(t-s)}}{\xi} \begin{pmatrix} \sin(\xi(t-s)) \\ \xi \cos(\xi(t-s)) - \lambda \sin(\xi(t-s)) \end{pmatrix} = \begin{pmatrix} \eta T_0 g_{t-s} \\ \eta T_0 h_{t-s} \end{pmatrix}.$$}

It follows that

$$x_t = \eta \int_0^t (T_0 - \alpha_0 y_s) g_{t-s} dB_s$$

and

$$y_t = \eta \int_0^t (T_0 - \alpha_0 y_s) h_{t-s} dB_s.$$}

14. Letting $Z_t = F(B_t)$, where $B_t = B^{(1)}_t + iB^{(2)}_t$, calculate

$$dZ_t = F_x(B_t) dB^{(1)}_t + F_y(B_t) dB^{(2)}_t + \frac{1}{2} F_{xx}(B_t) dB^{(1)}_t [B^{(1)}_t]_t + F_{xy}(B_t) dB^{(1)}_t [B^{(2)}_t]_t + \frac{1}{2} F_{yy}(B_t) dB^{(2)}_t [B^{(2)}_t]_t$$

$$= (u_x + iv_x) dB^{(1)}_t + (u_y + iv_y) dB^{(2)}_t + \frac{1}{2} (u_{xx} + iv_{xx} + u_{yy} + iv_{yy}) dt$$

$$= \langle F'(B_t), dB_t \rangle + \frac{1}{2} (v_{xy} - iu_{xy} + u_{yy} + iv_{yy}) dt$$

$$= \langle F'(B_t), dB_t \rangle + \frac{1}{2} (-u_{yy} - iv_{yy} + u_{yy} + iv_{yy}) dt$$

$$= \langle F'(B_t), dB_t \rangle.$$
15. Consider the non-linear stochastic differential equation

\[ dX_t = rX_t(K - X_t)\,dt + \beta X_t\,dB_t, \quad X_0 = x > 0. \]

Comparing to the deterministic Bernoulli equation, do a substitution \( Y_t = X_t^{-1} \), then

\[ dY_t = -rY_t(K - X_t)\,dt - \beta Y_t\,dB_t + \beta^2 Y_t\,dt \]
\[ = (-rK + \beta^2)Y_t\,dt - \beta Y_t\,dB_t + r\,dt. \]

Next do a new change of variables

\[ Z_t = Y_t e^{(rK - \beta^2)t} \]

and calculate

\[ dZ_t = -\beta Z_t\,dB_t + re^{(rK - \beta^2)t}\,dt \]
\[ \Rightarrow Z_t = e^{-\beta B_t} \left( x^{-1} + r \int_0^t e^{(rK - \beta^2)s + \beta B_s} \,ds \right). \]

Conclude that

\[ X_t = \frac{e^{(rK - \beta^2)t}Z_t}{x^{-1} + r \int_0^t e^{(rK - \beta^2)s + \beta B_s} \,ds}. \]

16. Consider the non-linear stochastic differential equation

\[ dX_t = f(t, X_t)\,dt + c(t)X_t\,dB_t, \quad X_0 = x. \]

(a) Let \( F_t(\omega) = \exp \left( - \int_0^t c(s)\,dB_s + \frac{1}{2} \int_0^t c(s)^2\,ds \right) \). Then calculate

\[ d(F_tX_t) = X_t\,dF_t + F_t\,dX_t + d[F_t, X_t] \]
\[ = X_t \left[ F_t \left( -c(t)\,dB_t - \frac{1}{2} c(t)^2\,dt - \frac{1}{2} c(t)^2\,dt \right) \right] \]
\[ + \left[ f(t, X_t)F_t\,dt + c(t)X_tF_t\,dB_t \right] - c(t)^2F_tX_t\,dt \]
\[ = f(t, X_t)F_t\,dt. \]

(b) Defining \( Y_t = F_tX_t \), deduce that

\[ \frac{dY_t}{dt} = F_t(\omega)f(t, F_t^{-1}(\omega)Y_t(\omega)) \]

(c) Consider \( dX_t = X_t^{-1} + \alpha X_t\,dB_t, X_0 = x > 0. \) Then

\[ \frac{dY_t}{dt} = e^{-2\alpha B_t + \alpha^2 t}Y_t^{-1}, \]
which implies
\[ Y_t = \sqrt{Y_0^2 + 2 \int_0^t e^{-2\alpha B_t + \alpha^2 s} ds} \]

and
\[ X_t = e^{\alpha B_t - \frac{\alpha^2 t}{2}} \sqrt{x^2 + 2 \int_0^t e^{-2\alpha B_t + \alpha^2 s} ds}. \]

(d) Consider \( dX_t = X_t^\gamma dt + \alpha X_t dB_t, X_0 = x > 0. \) Then
\[ \frac{dY_t}{dt} = e^{-(1-\gamma)B_t + (1-\gamma)\frac{\alpha^2 t}{2}} Y_t^\gamma, \]

which implies
\[ Y_t = \left(Y_0^{1-\gamma} + (1-\gamma) \int_0^t e^{-(1-\gamma)B_s + (1-\gamma)\frac{\alpha^2 s}{2}} ds\right)^{\frac{1}{1-\gamma}} \]

and
\[ X_t = e^{\alpha B_t - \frac{\alpha^2 t}{2}} \left(x^{1-\gamma} + (1-\gamma) \int_0^t e^{-(1-\gamma)B_s + (1-\gamma)\frac{\alpha^2 s}{2}} ds\right)^{\frac{1}{1-\gamma}}. \]

17. Let \( v \geq 0 \) satisfy \( v(t) \leq C + A \int_0^t v(s) ds \) and consider quantity \( w(t) = \int_0^t v(s) ds. \) Then
\[ w'(t) = v(t) \leq C + A \int_0^t v(s) ds = C + Aw(t). \]

Then for \( f(t) = w(t)e^{-At}, \) calculate
\[ f'(t) = e^{-At} (w'(t) - Aw(t)) \leq Ce^{-At} \]

and
\[ w(t)e^{-At} \leq \int_0^t Ce^{-As} ds = \frac{C}{A}(1 - e^{-At}) \]
\[ \implies w(t) \leq \frac{C}{A}(e^{At} - 1). \]

Deduce that
\[ v(t) \leq C + Aw(t) \leq Ce^{At}. \]
Chapter 6

The Filtering Problem
Chapter 7

Diffusions: Basic Properties
Chapter 8
Other Topics in Diffusion Theory
Chapter 9

Applications to Boundary Value Problems
Chapter 10

Applications to Optimal Stopping
Chapter 11

Applications to Stochastic Control
Chapter 12

Applications to Mathematical Finance