TWO PROBLEMS IN NON-LINEAR EVOLUTION EQUATIONS

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ABSTRACT

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Part I: The theory of coupling between internal and surface waves for stratified fluid domains is a rich source of dispersive and non-linear model equations with broad applications to ocean engineering. We study the two-dimensional water wave problem consisting of two fluid domains, the lower of which is infinitely deep, separated by a sharp interface, which is due in practice to a temperature or salinity gradient, and analyze the coupling effect of free internal and surface waves. Starting from the incompressible, irrotational Euler equations of motion for a two-layered fluid consisting of two different densities, we use its Hamiltonian formulation and the corresponding canonical variables to derive a coupled system for the evolution of two waves, where the small amplitude, internal long wave is modelled by a Benjamin-Ono equation. The surface elevation, on the other hand, has a shorter wavelength and is modelled by a modulated monochromatic wave whose envelope satisfies a time-dependent, linear Schrödinger equation. The coefficients of the coupled system are evaluated in terms of the physical parameters. Our results extend previous work on the coupled Korteweg-de-Vries and modulational regime for coupling

between internal and surface waves in shallow water by Craig, Guyenne and Sulem ([CGS11], [CGS12]) to the case of deep water.

Part II: Bochner formulas are often the starting point for the analysis of Riemannian manifolds with bounded Ricci curvature. We generalize the classical Bochner formula for the heat flow on evolving manifolds $(M, g_t)_{t\in[0,T]}$ to an infinite-dimensional Bochner formula for martingales on parabolic path space $P\mathcal{M}$ of space-time $\mathcal{M} = M \times [0,T]$. Our new Bochner formula and the inequalities that follow from it are strong enough to characterize solutions of the Ricci flow. Specifically, we obtain characterizations of the Ricci flow in terms of Bochner inequalities on parabolic path space. We also obtain gradient and Hessian estimates for martingales on parabolic path space, as well as condensed proofs of the prior characterizations of the Ricci flow from Haslhofer-Naber [HN18a]. Our results are parabolic counterparts of the recent results in the elliptic setting from [HN18b].

To my mother and father,
for your love and support.

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PUBLICATIONS

The content of Part I is based on a collaboration with Adilbek Kairzhan and Catherine Sulem, on which a paper is currently in preparation.

The content of Part II is presented in my paper, "A Bochner Formula on Path Space for the Ricci Flow" [Ken23], and has been published in *Calculus of Variations and Partial Differential Equations*. It is available at this link as well as on the arXiv.

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Part I

COUPLING BETWEEN INTERNAL AND SURFACE WAVES IN DEEP WATER

1 INTRODUCTION

A fluid domain, such as an ocean or a sea, is often stratified into layers of differing densities due to temperature or salinity gradients. Internal solitary waves and their effects on ocean dynamics have been widely observed with detection technology. In particular, these internal waves present implications for underwater navigation and the engineering of offshore structures. The signatures of these waves have been surveyed by hydrologists and oceanographers, such as Perry-Schimke [PS65], Osborne-Burch [OB80] and Helfrich-Melville [HM06].

Some early measurements made in the Andaman Sea found internal waves of high amplitude, 80 metres, and long wavelength, 2000 metres, with a thermocline situated at roughly 500-metres deep in the 1500-metre deep sea [PS65]. More recently, evidence of internal solitons inducing riptides in coastal seas has been observed [HM06]. A characteristic change in the reflectance of the water surface and the observed “ripple effect" have provided empirical evidence of coupling between a longer-scale internal wave and a shorter-scale, rougher and more rippled, surface wave. This striking phenomenon in which rough waters are present in a relatively quiescent sea has been described as the “mill pond effect" [OB80], which can be explained by the fact that water is calmer after the passage of internal waves.

According to the US Geological Survey (www.usgs.gov), if there is a sharp thermal gradient separating a warmer upper fluid domain from a cooler lower one, then the former will be marginally less dense for temperatures in excess of $4.4^{\circ}C$. As such, the density ratio is typically near unity and not less than 0.995 for upper temperatures below $32.2^{\circ}C$. A schematic of a two-dimensional fluid system consisting of two domains with a longer-scale internal wave and a shorter-scale wave envelope is shown below in Figure 1.1, in which there is effectively no bottom as we assume the lower fluid domain is infinitely deep. The conjugate variables η and η_1 denote the elevations of the internal and surface waves, while physical parameters g, h_1, ρ and ρ_1 are the gravitational constant, the height of the upper fluid domain and the two densities, respectively.

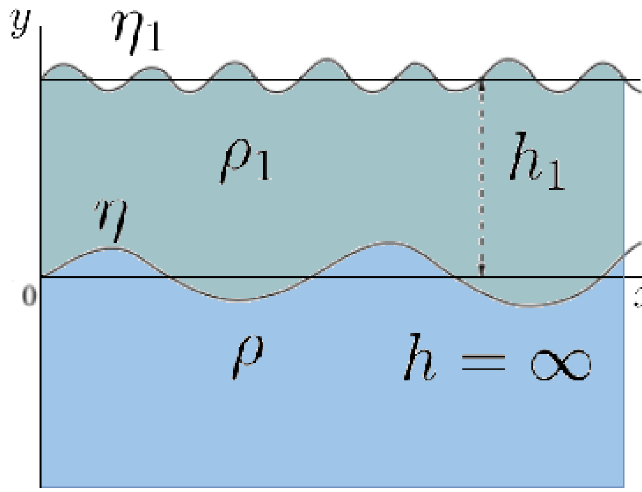

Figure 1.1: Two Fluids Separated by a Sharp Interface

There exists extensive literature on the resonant coupling between internal and free surface waves. For instance, there have been studies of resonance interaction for similar length scales by Gear-Grimshaw [GG84] and Părău-Dias [PD01], while differing length scales have been studied by Kawahara [Kaw72], Hashizume [Has80] and Funakoshi-Oikawa [FO83]. A system of two Korteweg-de-Vries (KdV) equations coupled by the strong resonant interaction of close phase velocities is produced in [GG84], while combination waves from the interaction of modes with the same phase speed but differing wavelengths are calculated in [PD01]. In the case of differing length scales ([Kaw72], [Has80] and [FO83]), steady solutions are studied through numerical computation. The model equations for non-linear interaction of the two modes have been derived in [Kaw72] and [Has80].

The phenomenon of surface rips and the mill pond effect after their passage have been described further by Craig-Guyenne-Sulem [CGS12]. In their paper, they address the characteristic narrowness of the surface rips in contrast to the broadening of internal solitary waves. Moreover, they precisely determine the surface wave, the location of its rip with respect to the centre of the internal wave as well as the degree to which the mill pond effect is due to wave breakage and the passage of internal waves.

From a mathematical formulation for two-layered water flows, we derive a coupled system of equations modelling the signature ripping and coupling effects induced by a non-linear internal wave on a lower-amplitude, modulating free surface, in the case of different length scales. We explicitly provide the non-linear coupling coefficients, which are dependent on the physical parameters of the system for the interaction including a density ratio between the two fluid domains that is close to unity. Specifically, we analyse the coupling effect of the internal and surface modes on the modulation of quasi-monochromatic surface waves caused by the resonant excitement when the group velocity of the surface wave coincides with the phase velocity of the longer-scale, less rippled, internal wave.

We consider a two-dimensional fluid domain consisting of two immiscible fluids separated by an interface, which idealises a sharp thermocline or pycnocline. We assume that large amplitude, non-linear and non-dispersive long-waves are generated at the interface. In practice, this occurs as tides move relatively cold water over the ridges of submerged mountains.

Starting from the incompressible, irrotational Euler equations of motion for two immiscible fluids in the close density regime of our water wave problem, we derive the Hamiltonian formulation, based on the original work of Zakharov's [Zak68] as well as work by Craig-Sulem [CS93] and Craig-Guyenne-Kalisch [CGK05], needed to perform both the necessary asymptotic analysis of the Dirichlet-Neumann operators and the normal mode analysis of the linearised equations.

Applying the Hamiltonian formulation of the water wave problem, in which the energy is a conserved quantity, and as done by Craig-Guyenne-Kalisch [CGK05], we write the quadratic and cubic terms of the Hamiltonian in canonically conjugate variables. Afterwards, we use multiple scale and modulation analysis to derive a higher-order Benjamin-Ono (BO) equation coupled to a linear Schrödinger equation that describe the time evolution of the internal wave and surface wave envelope, respectively. Under the Hamiltonian formulation, this problem has been studied extensively by Craig-Guyenne-Sulem ([CGS11], [CGS12] and [CGS15]) in the case of a shallow lower fluid. Our results extend their work to the case where the lower fluid has infinite depth.

The broad literature on internal waves in oceanography primarily focuses on two physical settings: (i) fixed lids and (ii) coupling between internal and surface waves. For the former case, a large class of scaling regimes have been used to model weak non-linearity of the interface, such as by Benjamin [Ben67], Ono [Ono75], Camassa-Choi [CC96] and Camassa-Choi-Michallet-Rusås-Sveen [CCM+06].

In this thesis, we focus on the latter case of coupled interaction between internal and surface waves in the deep water regime where we truncate the Hamiltonian at cubic as opposed to at quadratic order in the canonically conjugate variables. Given $\eta(x,t) \sim \varepsilon r(X,\tau)$, where η is the elevation of the internal wave, $X = \varepsilon x$ and $\tau = \varepsilon^2 t$, the free interface evolves according to a higher-order BO equation

$$\begin{aligned} \partial_\tau r = {} & \alpha_1 \partial_X(|D_X| r) + \alpha_2 \partial_X^3 r + \alpha_3 r(\partial_X r) \\ & + \alpha_4 \partial_X(|v_1|^2) + \alpha_5 \big[\partial_X(r(|D_X|r)) + |D_X|(r\partial_X r)\big] \\ & + \alpha_6 \partial_X\big[v_1(\overline{D_X v_1}) + \overline{v_1}(D_X v_1)\big] + \alpha_7 \partial_X\big(|D_X|(|v_1|^2)\big), \end{aligned} \tag{BO}$$

which is coupled to a free surface that both propagates at resonant group velocity $\omega'(k_0) = c_0$ and is modulated by a time-dependent, linear Schrödinger equation. Given $\eta_1 \sim \varepsilon^{1+\delta} v_1(X,\tau) e^{ik_0 x}$, where η_1 is the elevation of the surface wave, the surface envelope satisfies

$$i\partial_\tau v_1 = \beta_1 \partial_X^2 v_1 + \beta_2 r v_1 + \beta_3 i(\partial_X(r v_1) + r\partial_X v_1) + \beta_4 v_1(|D_X| r). \tag{LS}$$

The operators are $D_X := -i\partial_X$ and $|D_X| = D_X \mathrm{sgn}(D_X) = \mathcal{H}\partial_x$, where $\mathcal{H}$ is the Hilbert transform with symbol $-i \cdot \mathrm{sgn}(k)$. Each coefficient α_i and β_j, for $i \in \{1,2,3,4,5,6,7\}$ and $j \in \{1,2,3,4\}$, depends on the physical parameters: the gravitational constant g, the height of the upper fluid domain h_1 and the two densities ρ and ρ_1. We ultimately derive this coupled system of model equations (BO) and (LS) in terms of the physically-determined coefficients.

Part I of the thesis is organised as follows:

- In Chapter 2, we describe the water wave problem, beginning with Euler's equations for an incompressible, irrotational two-layered fluid. We present its Hamiltonian formulation in terms of canonical variables $(\eta, \eta_1, \xi, \xi_1)$ where η, η_1 are the elevations of the internal and surface waves and ξ, ξ_1 are their conjugated variables.

- In Chapter 3, we recall from [CS93] and [CGK05] the expansions of the Dirichlet-Neumann operators for the lower and upper fluid domains in powers of variables η and η_1.

- In Chapter 4, we present the linear analysis of the fluids near rest, find the dispersion relation and perform a normal mode decomposition to diagonalise the quadratic part of the Hamiltonian.

- In Chapter 5, we calculate the cubic terms of the Hamiltonian in two canonically conjugate coordinate systems.

- In Chapter 6, we introduce the scaling regime under consideration. The internal wave varies on long scales and has small amplitude as described by the Benjamin-Ono scaling $\eta \sim \varepsilon\tilde{\eta}(\varepsilon x)$. On the other hand, the upper surface is modelled by a modulated wave packet $\eta_1 \sim \varepsilon_1\tilde{\eta}_1(\varepsilon x)e^{ik_0 x}$, where we assume $\varepsilon_1 \ll \varepsilon$.

- In Chapter 7, we use Hamiltonian transformation theory and a resonant condition to derive an asymptotic system composed of a higher-order Benjamin-Ono equation for the internal wave coupled to a linear Schrödinger equation for the wave envelope of the free surface. We give the coefficients of this system explicitly in terms of the physical parameters. We conclude with a brief discussion of future work.

2 FORMULATION OF THE PROBLEM

2.1 EQUATIONS OF MOTION

The two-dimensional fluid domain is composed of two immiscible fluids separated by a sharp free interface $\{y = \eta(x)\}$ into lower and upper regions given by

$$S(\eta) = \{(x,y), x \in \mathbb{R}, -\infty < y < \eta(x,t)\}, \tag{2.1}$$

with lower fluid density ρ, and

$$S_1(\eta, \eta_1) = \{(x,y), x \in \mathbb{R}, \eta(x,t) < y < h_1 + \eta_1(x,t)\}, \tag{2.2}$$

with upper fluid density ρ_1, respectively. We assume that the system is stably configured, $\rho > \rho_1$, and that the fluid motion is a potential flow, namely that the velocities $\mathbf{u}(x,y,t) = \nabla\phi(x,y,t)$ in $S(t;\eta)$ and $\mathbf{u}_1(x,y,t) = \nabla\phi_1(x,y,t)$ in $S(t;\eta,\eta_1)$ with the two velocity potentials, ϕ and ϕ_1, satisfying

$$\begin{cases} \Delta\phi &= 0, \text{ in } S(t;\eta) \\ \Delta\phi_1 &= 0, \text{ in } S_1(t;\eta,\eta_1). \end{cases} \tag{2.3}$$

First we assume that velocity flow $\phi(x,y) \to 0$ as $y \to -\infty$. We shall also assume boundary conditions, governed by kinematics and the balance of forces, for both the free interface and the free surface.

On the interface between the two fluid domains, we impose kinematic and physical constraints. Letting $\hat{\nu}$ refer to the exterior unit normal pointing out of the free interface, the two equations addressing the kinematics on the interface are

$$\partial_t\eta = \partial_y\phi - (\partial_x\eta)(\partial_x\phi) = \nabla\phi \cdot \hat{\nu}\sqrt{1 + |\partial_x\eta|^2} \tag{2.4}$$

and

$$\partial_t \eta = \partial_y \phi_1 - (\partial_x \eta)(\partial_x \phi_1) = \nabla \phi_1 \cdot \hat{\nu} \left(-\sqrt{1 + |\partial_x \eta|^2} \right). \tag{2.5}$$

On the other hand, the Bernoulli condition imposes a physical constraint

$$\rho \left(\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g\eta \right) = \rho_1 \left(\partial_t \phi_1 + \frac{1}{2} |\nabla \phi_1|^2 + g\eta \right). \tag{2.6}$$

Finally, on the upper free surface $\{y = \eta_1(x) + h_1\}$, velocity potential, ϕ_1, and η_1 satisfy both a kinematic condition

$$\partial_t \eta_1 = \partial_y \phi_1 - (\partial_x \eta_1)(\partial_x \phi_1) = \nabla \phi_1 \cdot \hat{\nu_1} \sqrt{1 + (\partial_x \eta)^2}, \tag{2.7}$$

where $\hat{\nu_1}$ be the exterior unit normal pointing out of the top of the free interface, and a Bernoulli condition

$$\partial_t \phi_1 + \frac{1}{2} |\nabla \phi_1|^2 + g\eta_1 = 0. \tag{2.8}$$

Our goal will be to describe the coupled evolution of the free interface and the free surface.

2.2 CANONICAL VARIABLES AND HAMILTONIAN FORMULATION

The water wave problem has a Hamiltonian formulation as described in Benjamin-Bridges [BB97] and Craig-Sulem [CGK05]. See also Zakharov [Zak68] and Craig-Sulem [CS93] for further details. The strategy for solving the coupled problem of free internal and surface waves is to use the Lagrangian formulation, which depends on the perturbations both of the free interface $\eta(x,t)$ and the free surface $\eta_1(x,t)$. Then we derive classically canonical variables to provide a Hamiltonian formulation of the problem.

The kinetic energy is the weighted sum of the gradients of the two potentials of the velocity flows

$$K = \frac{1}{2} \int_{\mathbb{R}} \int_{-\infty}^{\eta(x)} \rho |\nabla \phi(x,y)|^2 \, dy \, dx + \frac{1}{2} \int_{\mathbb{R}} \int_{\eta(x)}^{h_1 + \eta_1(x)} \rho_1 |\nabla \phi_1(x,y)|^2 \, dy \, dx, \tag{2.9}$$

while the potential energy is given by

$$V = \frac{1}{2}\int_{\mathbb{R}} g(\rho - \rho_1)\eta^2(x)\,dx + \frac{1}{2}\int_{\mathbb{R}} g\rho_1\left((h_1+\eta_1)^2(x) - h_1^2\right)\,dx. \tag{2.10}$$

In analogy with Lagrangian mechanics, (η, η_1) are the spatial coordinates. We will reformulate the problem in terms of canonically conjugate variables to provide a Hamiltonian formulation of the problem.

We express the kinetic energy in terms of the boundary values for the two velocity potentials and two Dirichlet-Neumann operators. We denote the traces of the velocity potentials on the boundaries of the fluid domain by

$$\begin{cases} \Phi(x) &= \phi(x, \eta(x)) \\ \Phi_1(x) &= \phi_1(x, \eta(x)) \\ \Phi_2(x) &= \phi_1(x, h_1 + \eta_1(x)). \end{cases} \tag{2.11}$$

2.2.1 *Dirichlet-Neumann Operators*

We define Dirichlet-Neumann operators for the internal and surface waves in the following way.

The Dirichlet-Neumann operator for the lower fluid domain is given by

$$G(\eta)\Phi(x) = ((\nabla\phi)\cdot\hat{\nu})(x, \eta(x))\sqrt{1 + |\partial_x\eta|^2}, \tag{2.12}$$

where again $\hat{\nu}$ refers to the exterior unit normal pointing out of the free interface.

Due to the coupling of data $\Phi_1(x) = \phi_1(x, \eta(x))$ and $\Phi_2(x) = \phi_1(x, h_1 + \eta_1(x))$, the Dirichlet-Neumann operator, which measures velocity flux, is defined as a matrix operator

$$\begin{aligned} &\begin{pmatrix} G_{11}(\eta,\eta_1) & G_{12}(\eta,\eta_1) \\ G_{21}(\eta,\eta_1) & G_{22}(\eta,\eta_1) \end{pmatrix} \begin{pmatrix} \Phi_1(x) \\ \Phi_2(x) \end{pmatrix} \\ &\quad = \begin{pmatrix} -(\nabla\phi_1\cdot\hat{\nu})(x,\eta(x))\sqrt{1+|\partial_x\eta|^2} \\ (\nabla\phi_1\cdot\hat{\nu_1})(x, h_1+\eta_1(x))\sqrt{1+|\partial_x\eta_1|^2} \end{pmatrix} \end{aligned} \tag{2.13}$$

where again $\hat{\nu_1}$ is the exterior unit normal pointing out of the top of the free surface.

2.2.2 *The Kinetic Energy*

Using Green's identities and rewriting the normal derivatives of the traces of the velocity in terms of the Dirichlet-Neumann operators, we express the kinetic energy of the system as

$$\begin{aligned} K &= \frac{1}{2}\int_{\mathbb{R}} \rho \Phi G(\eta)\Phi \, dx \\ &+ \frac{1}{2}\int_{\mathbb{R}} \rho_1 \begin{pmatrix} \Phi_1 & \Phi_2 \end{pmatrix} \begin{pmatrix} G_{11}(\eta,\eta_1) & G_{12}(\eta,\eta_1) \\ G_{21}(\eta,\eta_1) & G_{22}(\eta,\eta_1) \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} dx. \end{aligned} \tag{2.14}$$

Continuing in the Lagrangian framework, we introduce velocity coordinates $(\dot{\eta}, \dot{\eta}_1)$ that are orthogonal to the spatial coordinates. In terms of the Dirichlet-Neumann operators and the boundary conditions, we define these variables by

$$\begin{cases} \dot{\eta} &= G(\eta)\Phi = -(G_{11}(\eta,\eta_1)\Phi_1 + G_{12}(\eta,\eta_1)\Phi_2) \\ \dot{\eta}_1 &= G_{21}(\eta,\eta_1)\Phi_1 + G_{22}(\eta,\eta_1)\Phi_2. \end{cases} \tag{2.15}$$

We write the Lagrangian for our coupled water wave problem based on Equations (2.10) and (2.14) in terms of variables $(\eta, \eta_1, \dot{\eta}, \dot{\eta}_1)$

$$\begin{aligned} L :&= K - V \\ &= \frac{1}{2}\int_{\mathbb{R}} \rho\Phi G(\eta)\Phi\, dx + \frac{1}{2}\int_{\mathbb{R}} \rho_1 \begin{pmatrix} \Phi_1 & \Phi_2 \end{pmatrix} \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} dx \\ &\quad - \frac{1}{2}\int_{\mathbb{R}} g(\rho-\rho_1)\eta^2(x)\, dx - \frac{1}{2}\int_{\mathbb{R}} g\rho_1 \left((h_1+\eta_1)^2(x) - h_1^2\right) dx \\ &= \frac{1}{2}\int_{\mathbb{R}} \rho\dot{\eta} G^{-1}(\eta)\dot{\eta}\, dx + \frac{1}{2}\int_{\mathbb{R}} \rho_1 \begin{pmatrix} \dot{\eta} & \dot{\eta}_1 \end{pmatrix} \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}^{-1} \begin{pmatrix} \dot{\eta} \\ \dot{\eta}_1 \end{pmatrix} dx \\ &\quad - \frac{1}{2}\int_{\mathbb{R}} g(\rho-\rho_1)\eta^2(x)\, dx - \frac{1}{2}\int_{\mathbb{R}} g\rho_1 \left((h_1+\eta_1(x))^2 - h_1^2\right) dx. \end{aligned} \tag{2.16}$$

Next we compute the canonically conjugate variables by taking the Legendre transform

$$\begin{aligned} \begin{pmatrix} \xi \\ \xi_1 \end{pmatrix} = \begin{pmatrix} \delta_{\dot{\eta}} L \\ \delta_{\dot{\eta}_1} L \end{pmatrix} &= \rho \begin{pmatrix} G^{-1}(\eta)\dot{\eta} \\ 0 \end{pmatrix} + \rho_1 \begin{pmatrix} G_{11} & -G_{12} \\ -G_{21} & G_{22} \end{pmatrix}^{-1} \begin{pmatrix} \dot{\eta} \\ \dot{\eta}_1 \end{pmatrix} \\ &= \begin{pmatrix} \rho\Phi - \rho_1\Phi_1 \\ \rho_1\Phi_2 \end{pmatrix}. \end{aligned} \tag{2.17}$$

This is also shown in Benjamin and Bridges [BB97] while the classical result is found in Landau and Lifschitz [LL60]. Using these canonically conjugate variables, we provide a canonical Hamiltonian description of the problem.

Theorem 2.1 ([CGK05])**.** *The system of equations possesses a canonical Hamiltonian structure in terms of the canonically conjugate coordinates, $\eta(x,t)$, $\eta_1(x,t)$, $\xi(x,t)$ and $\xi_1(x,t)$, with Hamiltonian H given by the conserved energy.*

We restate the kinetic energy in these coordinates

$$K = \frac{1}{2}\int_{\mathbb{R}} \begin{pmatrix}\xi & \xi_1\end{pmatrix} \begin{pmatrix}\dot\eta \\ \dot\eta_1\end{pmatrix} dx = \frac{1}{2}\int_{\mathbb{R}} \begin{pmatrix}\xi & \xi_1\end{pmatrix} \begin{pmatrix}-G_{11} & -G_{12} \\ G_{21} & G_{22}\end{pmatrix} \begin{pmatrix}\Phi_1 \\ \Phi_2\end{pmatrix} dx. \tag{2.18}$$

Defining

$$B(\eta,\eta_1) := \rho G_{11}(\eta,\eta_1) + \rho_1 G(\eta), \tag{2.19}$$

we compute

$$\begin{aligned} B\Phi &= \rho G_{11}\Phi + \rho_1 G(\eta)\Phi \\ &= \rho G_{11}\Phi - \rho_1(G_{11}\Phi_1 + G_{12}\Phi_2) \\ &= G_{11}(\rho\Phi - \rho_1\Phi_1) - G_{12}\rho_1\Phi_2 \\ &= G_{11}\xi - G_{12}\xi_1. \end{aligned} \tag{2.20}$$

Similar calculations, such as in Benjamin-Bridges [BB97], yield

$$\begin{cases} B\Phi_1 & = -G(\eta)\xi - \frac{\rho}{\rho_1} G_{12}\xi_1 \\ \rho_1\Phi_2 & = \xi_1 \end{cases} \tag{2.21}$$

and the kinetic energy can be rewritten as

$$\begin{aligned} K &= \frac{1}{2}\int_{\mathbb{R}} \begin{pmatrix}\xi & \xi_1\end{pmatrix} \begin{pmatrix}\dot\eta \\ \dot\eta_1\end{pmatrix} dx \\ &= \frac{1}{2}\int_{\mathbb{R}} \begin{pmatrix}\xi & \xi_1\end{pmatrix} \begin{pmatrix} G_{11}B^{-1}G(\eta) & -G(\eta)B^{-1}G_{12} \\ -G_{21}B^{-1}G(\eta) & \rho^{-1}G_{22} - \rho\rho_1^{-1}G_{21}B^{-1}G_{12} \end{pmatrix} \begin{pmatrix}\xi \\ \xi_1\end{pmatrix} dx. \end{aligned} \tag{2.22}$$

2.2.3 *The Hamiltonian and Equations of Motion*

Hamilton's equations

$$\partial_t \begin{pmatrix} \eta \\ \xi \\ \eta_1 \\ \xi_1 \end{pmatrix} \equiv J\nabla H = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_\eta H \\ \delta_\xi H \\ \delta_{\eta_1} H \\ \delta_{\xi_1} H \end{pmatrix} \tag{2.23}$$

describe the motion of both the free interface and free surface, while the Hamiltonian is

$$\begin{aligned} H &= K + V \\ &= \frac{1}{2}\int_{\mathbb{R}} \begin{pmatrix} \xi & \xi_1 \end{pmatrix} \begin{pmatrix} G_{11}B^{-1}G(\eta) & -G(\eta)B^{-1}G_{12} \\ -G_{21}B^{-1}G(\eta) & \rho^{-1}G_{22} - \rho\rho_1^{-1}G_{21}B^{-1}G_{12} \end{pmatrix} \begin{pmatrix} \xi \\ \xi_1 \end{pmatrix} dx \\ &\quad + \frac{1}{2}\int_{\mathbb{R}} g(\rho - \rho_1)\eta^2(x)\, dx + \frac{1}{2}\int_{\mathbb{R}} g\rho_1 \left((h_1 + \eta_1)^2(x) - h_1^2\right) dx, \end{aligned} \tag{2.24}$$

written in terms of our canonical variables $(\eta, \eta_1, \xi, \xi_1)$.

3

DIRICHLET-NEUMANN OPERATORS

We expand the Dirichlet-Neumann operators, $G(\eta)$ and $\mathbf{G}(\eta, \eta_1)$, for the lower and upper fluid domains $S(\eta)$ and $S_1(\eta; \eta_1)$, respectively, in terms of powers of η and η_1. We find the linear expansion of the Dirichlet-Neumann operators for the free interface $G(\eta)$

$$G(\eta) = G^{(0)} + G^{(1)}(\eta) + \mathcal{O}(\eta^2) \tag{3.1}$$

as well as the free surface $\mathbf{G}(\eta, \eta_1) = \big(G_{ij}(\eta, \eta_1)\big)$

$$G_{ij}(\eta, \eta_1) = G_{ij}^{(0)} + G_{ij}^{(10)}(\eta) + G_{ij}^{(01)}(\eta_1) + \mathcal{O}(|(\eta, \eta_1)|^2), \tag{3.2}$$

where $G_{ij}^{(10)}$ and $G_{ij}^{(01)}$ refer to the linear dependencies in η and η_1, respectively, for $i, j \in \{1, 2\}$. These expansions will be shown in the following propositions, which reproduce the results for $G(\eta)$, as found in Craig-Sulem [CS93], and $G(\eta, \eta_1)$, as found in Craig-Guyenne-Kalisch [CGK05], respectively. The analyticity of the series and its convergence for small data is proved by Coifman and Meyer [CM85]. Also see Craig-Schanz-Sulem [CSS97] and Lannes [Lan13] for further details.

3.1 TAYLOR EXPANSIONS IN THE LOWER FLUID IN POWERS OF η

Proposition 3.1. [CS93] The Dirichlet-Neumann operator for the lower fluid domain $S(\eta)$ is

$$G(\eta) = |D| + D\eta D - |D|\eta|D| + \mathcal{O}(\eta^2), \tag{3.3}$$

where $D = -i\partial_x$.

Proof. For the lower fluid domain $S(\eta)$, a particular basis of harmonic functions is given by

$$\phi_k(x, y) = \left(a(k)e^{ky} + b(k)e^{-ky}\right) e^{ikx}, \tag{3.4}$$

where $a(k) = \mathbb{1}_{k>0}(k)$ and $b(k) = \mathbb{1}_{k<0}(k)$ with normalization $\phi_k(x,0) = e^{ikx}$. Using Taylor expansion, the boundary values are given by the trace on the free interface

$$\Phi_k(x) = \phi_k(x,\eta(x)) = \sum_{j\geqslant 0} \frac{(k\eta(x))^j}{j!}\left(a(k) + (-1)^j b(k)\right) e^{ikx}. \tag{3.5}$$

Next we relate the normal derivative on the free interface to $G(\eta)\phi_k$

$$\begin{aligned} G(\eta)\Phi_k(x) &= (\nabla\phi\cdot\hat{v})(x,\eta(x))\sqrt{1+|\partial_x\eta|^2}\Big|_{y=\eta(x)} \\ &= \sum_{j\geqslant 0}\frac{\eta(x)^j}{j!}(-\partial_x\eta(x))(ik^{j+1})\left(a(k)+(-1)^j b(k)\right)e^{ikx} \\ &\quad + \sum_{j\geqslant 0}\frac{\eta(x)^j}{j!}(k^{j+1})\left(a(k)+(-1)^{j+1}b(k)\right)e^{ikx}. \end{aligned} \tag{3.6}$$

to the Taylor expansions $G^{(j)}(\eta)\Phi_k(x)$. The constant term is given by

$$G^{(0)}(\eta)e^{ikx} = k\left(a(k) - b(k)\right)e^{ikx} = k\mathrm{sgn}(k)e^{ikx} = |k|e^{ikx} \tag{3.7}$$

and thus

$$G^{(0)} = |D|, \tag{3.8}$$

where the Dirichlet-Neumann operator $G^{(0)}$ is written as a Fourier multiplier in terms of operator D. Rewriting the higher-order terms of the Taylor expansion,

$$\begin{aligned} G^{(j)}(\eta) &= \frac{1}{j!}D\eta^j(x)D^j\left(a(D)+(-1)^{j+1}b(D)\right) \\ &\quad - \sum_{\ell=1}^{j} G^{(j-\ell)}(\eta)\frac{1}{\ell!}\eta^\ell(x)D^\ell\left(a(D)+(-1)^\ell b(D)\right), \end{aligned} \tag{3.9}$$

we can then read the first-order term of the Dirichlet-Neumann operator $G(\eta)$

$$G^{(1)}(\eta) = D\eta D - G^{(0)}\eta D\mathrm{sgn}(D) = D\eta D - |D|\eta|D|, \tag{3.10}$$

as well as higher-order terms recursively, such as found in [CS93]. □

3.2 TAYLOR EXPANSIONS IN THE UPPER FLUID IN POWERS OF (η, η_1)

Proposition 3.2 ([CGK05])**.** The Dirichlet-Neumann operator for the upper fluid domain $S(\eta;\eta_1)$ is

$$\mathbf{G}(\eta,\eta_1) = \mathbf{G}^{(0)} + \mathbf{G}^{(10)}(\eta) + \mathbf{G}^{(01)}(\eta_1) + \mathcal{O}(|(\eta,\eta_1)|^2), \tag{3.11}$$

where

$$\mathbf{G}^{(0)} := \begin{pmatrix} D\coth(h_1 D) & -D\operatorname{csch}(h_1 D) \\ -D\operatorname{csch}(h_1 D) & D\coth(h_1 D) \end{pmatrix}, \tag{3.12}$$

$$\mathbf{G}^{(10)}(\eta) := \begin{pmatrix} D\coth(h_1 D)\eta D\coth(h_1 D) - D\eta D & -D\coth(h_1 D)\eta D\operatorname{csch}(h_1 D) \\ -D\operatorname{csch}(h_1 D)\eta D\coth(h_1 D) & D\operatorname{csch}(h_1 D)\eta\operatorname{csch}(h_1 D) \end{pmatrix} \tag{3.13}$$

and

$$\mathbf{G}^{(01)}(\eta_1) := \begin{pmatrix} -D\operatorname{csch}(h_1 D)\eta_1 D\operatorname{csch}(h_1 D) & D\operatorname{csch}(h_1 D)\eta_1 D\coth(h_1 D) \\ D\coth(h_1 D)\eta_1 D\operatorname{csch}(h_1 D) & D\eta_1 D - D\coth(h_1 D)\eta_1 D\coth(h_1 D) \end{pmatrix}. \tag{3.14}$$

Proof. For the upper fluid domain $S(\eta;\eta_1)$, we also consider a particular basis of harmonic functions

$$\phi_{1,k}(x,y) = (a(k)e^{ky} + b(k)e^{-ky})e^{ikx} \tag{3.15}$$

satisfying boundary conditions

$$\begin{cases} \Phi_{1,k}(x) = \phi_{1,k}(x,\eta(x)) = (a(k)e^{k\eta(x)} + b(k)e^{-k\eta(x)})e^{ikx} \\ \Phi_{2,k}(x) = \phi_{1,k}(x,h_1+\eta_1(x)) = (a(k)e^{kh_1}e^{k\eta_1(x)} + b(k)e^{-kh_1}e^{-k\eta_1(x)})e^{ikx}. \end{cases} \tag{3.16}$$

As before, we calculate the normal derivatives on the free interface

$$\begin{aligned} &G_{11}(\eta,\eta_1)\Phi_{1,k} + G_{12}(\eta,\eta_1)\Phi_{2,k} \\ &\quad = -(\nabla\phi_{1,k}\cdot\hat{\nu})(1+|\partial_x\eta(x)|^2)^{1/2}\big|_{y=\eta(x)} \\ &\quad = \sum_{j\geqslant 0}\frac{e^{ikx}}{j!}\eta(x)^j(\partial_x\eta(x))(ik^{j+1})\left(a(k)+(-1)^j b(k)\right) \\ &\qquad - \sum_{j\geqslant 0}\frac{e^{ikx}}{j!}\eta(x)^j(k^{j+1})\left(a(k)+(-1)^{j+1}b(k)\right) \end{aligned} \tag{3.17}$$

as well as those on the free surface

$$\begin{aligned} &G_{21}(\eta,\eta_1)\Phi_{1,k} + G_{22}(\eta,\eta_1)\Phi_{2,k} \\ &\quad = (\nabla\phi_{1,k}\cdot\hat{\nu_1})(1+|\partial_x\eta_1(x)|^2)^{1/2}\nu_1\big|_{y=h_1+\eta_1(x)} \\ &\quad = \sum_{j\geqslant 0}\frac{e^{ikx}}{j!}\eta_1(x)^j(-\partial_x\eta_1(x))(ik^{j+1})\left(a(k)e^{h_1k}+(-1)^j b(k)e^{-h_1k}\right) \\ &\qquad + \sum_{j\geqslant 0}\frac{e^{ikx}}{j!}\eta_1(x)^j(k^{j+1})\left(a(k)e^{h_1k}+(-1)^{j+1}b(k)e^{-h_1k}\right). \end{aligned} \tag{3.18}$$

Taking a basis of harmonic functions with coefficients of the form

$$(a_1(k), b_1(k)) = \left(\frac{-e^{-h_1k}}{e^{h_1k}-e^{-h_1k}}, \frac{e^{h_1k}}{e^{h_1k}-e^{-h_1k}}\right) \tag{3.19}$$

and

$$(a_2(k), b_2(k)) = \left(\frac{1}{e^{h_1k}-e^{-h_1k}}, \frac{-1}{e^{h_1k}-e^{-h_1k}}\right), \tag{3.20}$$

we relate these normal derivatives as before by admitting double Taylor expansions of the Dirichlet-Neumann operator in terms of η and η_1.

Now reading the Taylor expansion recursively, using the first basis of harmonic functions, we find two of the constant terms

$$\begin{aligned} \begin{pmatrix} G_{11}^{(0)} & G_{12}^{(0)} \\ G_{21}^{(0)} & G_{22}^{(0)} \end{pmatrix}\begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix} &= \begin{pmatrix} -k(a_1(k)-b_1(k))e^{ikx} \\ k(a_1(k)e^{h_1k}-b_1(k)e^{-h_1k})e^{ikx} \end{pmatrix} \\ &= \begin{pmatrix} k\coth(h_1k)e^{ikx} \\ -k\operatorname{csch}(h_1k)e^{ikx} \end{pmatrix}, \end{aligned} \tag{3.21}$$

whence

$$G_{11}^{(0)}(D) = D\text{coth}(h_1 D)\,; \quad G_{12}^{(0)}(D) = -D\text{csch}(h_1 D). \tag{3.22}$$

Similarly, we find the remaining two constant terms

$$\begin{aligned} \begin{pmatrix} G_{11}^{(0)} & G_{12}^{(0)} \\ G_{21}^{(0)} & G_{22}^{(0)} \end{pmatrix} \begin{pmatrix} 0 \\ e^{ikx} \end{pmatrix} &= \begin{pmatrix} -k(a_2(k) - b_2(k))e^{ikx} \\ k(a_2(k)e^{h_1 k} - b_2(k)e^{-h_1 k})e^{ikx} \end{pmatrix} \\ &= \begin{pmatrix} -k\text{csch}(h_1 k)e^{ikx} \\ k\text{coth}(h_1 k)e^{ikx} \end{pmatrix}, \end{aligned} \tag{3.23}$$

whence

$$G_{21}^{(0)}(D) = -D\text{csch}(h_1 D)\,; \quad G_{22}^{(0)}(D) = D\text{coth}(h_1 D). \tag{3.24}$$

By a similar argument to the previous proposition, we can read recursively the first-order (as well as higher-order) terms, the expansion of which is presented in [CGK05] and [CGS12]. □

Using Propositions 3.1 and 3.2, we can verify

$$\begin{cases} G_{12}^{(10)} = G_{11}^{(0)} \eta G_{12}^{(0)}; & G_{12}^{(01)} = -G_{12}^{(0)} \eta_1 G_{11}^{(0)} \\ G_{21}^{(10)} = G_{12}^{(0)} \eta G_{11}^{(0)}; & G_{21}^{(01)} = -G_{11}^{(0)} \eta_1 G_{12}^{(0)} \\ G_{22}^{(10)} = G_{12}^{(0)} \eta G_{12}^{(0)}; & G_{22}^{(01)} = -G_{11}^{(0)} \eta_1 G_{11}^{(0)} + D\eta_1 D. \end{cases} \tag{3.25}$$

Moreover, the operator $B(\eta, \eta_1)$ defined in Equation (2.19) has the Taylor expansion in (η, η_1) of the form

$$B = B_0 + B^{(1)} + \mathcal{O}(|(\eta, \eta_1)|^2) \tag{3.26}$$

where

$$B_0 := \rho G_{11}^{(0)} + \rho_1 G^{(0)}, \qquad B^{(1)} := \rho G_{11}^{(10)} + \rho G_{11}^{(01)} + \rho_1 G^{(1)} \tag{3.27}$$

and we intentionally write B_0 instead of $B^{(0)}$ to simplify further notations. We also write the inverse of the operator B^{-1}

$$\begin{aligned} B^{-1} &= \left(B_0 + B^{(1)} + \mathcal{O}(|(\eta, \eta_1)|^2)\right)^{-1} \\ &= B_0^{-1} - B_0^{-1} B^{(1)} B_0^{-1} + \mathcal{O}(|(\eta, \eta_1)|^2). \end{aligned} \tag{3.28}$$

These calculations will be used to perform the linear analysis and find the cubic terms of the Hamiltonian.

4

LINEAR ANALYSIS

4.1 LINEARISED EQUATIONS NEAR FLUIDS AT REST

We derive the quadratic Hamiltonian $H^{(2)} = K^{(2)} + V^{(2)}$ and then write the linearised equations of motion near $\eta = \eta_1 = 0$. We recall the Hamiltonian

$$H = \frac{1}{2}\int_{\mathbb{R}} (\xi \quad \xi_1) \begin{pmatrix} G_{11}B^{-1}G(\eta) & -G(\eta)B^{-1}G_{12} \\ -G_{21}B^{-1}G(\eta) & \rho_1^{-1}G_{22} - \rho\rho_1^{-1}G_{21}B^{-1}G_{12} \end{pmatrix} \begin{pmatrix} \xi \\ \xi_1 \end{pmatrix} dx$$
$$+ \frac{1}{2}\int_{\mathbb{R}} g(\rho - \rho_1)\eta^2(x)\, dx + \frac{1}{2}\int_{\mathbb{R}} g\rho_1 \left((h_1 + \eta_1)^2(x) - h_1^2\right) dx \tag{4.1}$$

from the previous section. First we note that there is no linear term in the Hamiltonian since h_1 can be chosen to that $\int_{\mathbb{R}} \eta_1(x)\, dx$, the net volume above h_1, is zero. Next, we proceed to the quadratic Hamiltonian in canonical variables using the lowest-order terms in the Taylor expansions of the Dirichlet-Neumann operators.

Proposition 4.1. In canonical variables $(\eta, \eta_1, \xi, \xi_1)$, the quadratic part of the Hamiltonian is

$$H^{(2)} = \frac{1}{2}\int_{\mathbb{R}} [\xi G_{11}^{(0)} G^{(0)} B_0^{-1}\xi - 2\xi G^{(0)} B_0^{-1} G_{12}^{(0)} \xi_1$$
$$+ \xi_1 \rho_1^{-1} G^{(0)} (\rho_1 G_{11}^{(0)} + \rho G^{(0)}) B_0^{-1} \xi_1] + [g(\rho - \rho_1)\eta^2 + g\rho_1\eta_1^2]\, dx, \tag{4.2}$$

where $B_0 = \rho D\coth(h_1 D) + \rho_1 |D|$ from Equation (3.27).

Proof. We have the quadratic part of the potential energy

$$V^{(2)} = \frac{1}{2}\int_{\mathbb{R}} [g(\rho - \rho_1)\eta^2 + g\rho_1\eta_1^2]\, dx \tag{4.3}$$

and that of the kinetic energy

$$K^{(2)} = \frac{1}{2}\int [\xi G_{11}^{(0)} B_0^{-1} G^{(0)} \xi - 2\xi G^{(0)} G_{12}^{(0)} B_0^{-1} \xi_1$$
$$+ \xi_1 \left(\rho_1^{-1} G_{22}^{(0)} - \rho\rho_1^{-1} B_0^{-1} (G_{12}^{(0)})^2\right) \xi_1]\, dx. \tag{4.4}$$

Using the identity $(G_{11}^{(0)})^2 - (G_{12}^{(0)})^2 = (G^{(0)})^2$, the third term can be simplified to

$$\begin{aligned}\rho_1^{-1}G_{22}^{(0)} - \rho\rho_1^{-1}B_0^{-1}(G_{12}^{(0)})^2 &= \rho_1^{-1}\left(G_{11}^{(0)}(\rho G_{11}^{(0)} + \rho_1 G^{(0)}) - \rho(G_{12}^{(0)})^2\right)B_0^{-1} \\ &= \rho_1^{-1}G^{(0)}(\rho_1 G_{11}^{(0)} + \rho G^{(0)})B_0^{-1}, \end{aligned} \tag{4.5}$$

which yields the result. □

The linearised equations of motion can now be written.

Proposition 4.2. The linearised equations of motion for (η, ξ) are given by

$$\begin{cases} \partial_t \eta = \delta_\xi H^{(2)} = G_{11}^{(0)} B_0^{-1} G^{(0)} \xi - G^{(0)} G_{12}^{(0)} B_0^{-1} \xi_1 \\ \partial_t \xi = -\delta_\eta H^{(2)} = -g(\rho - \rho_1)\eta, \end{cases} \tag{4.6}$$

while those for (η_1, ξ_1) are given by

$$\begin{cases} \partial_t \eta_1 = \delta_{\xi_1} H^{(2)} = -G^{(0)} G_{12}^{(0)} B_0^{-1} \xi + \left(\rho_1^{-1} G_{22}^{(0)} - \rho\rho_1^{-1} B_0^{-1} (G_{12}^{(0)})^2\right) \xi_1 \\ \partial_t \xi_1 = -\delta_{\eta_1} H^{(2)} = -g\rho_1 \eta_1. \end{cases} \tag{4.7}$$

Proof. The linearised equations of motion follow directly from applying Hamilton's equations. For example, we can calculate the functional derivatives for canonical variables ξ

$$\begin{aligned}\delta_\xi H^{(2)}[\xi, \xi_1, \eta, \eta_1](v) &= \lim_{\delta \to 0^+} \delta^{-1}\left(H^{(2)}[\xi + \delta v] - H^{(2)}[\xi]\right) \\ &= \left(G_{11}^{(0)} B_0^{-1} G^{(0)}(\eta)\xi + G^{(0)} G_{12}^{(0)} B_0^{-1} \xi_1, v\right). \end{aligned} \tag{4.8}$$

The calculations for the remaining variables are similar. □

4.2 DISPERSION RELATION

We derive the dispersion relation for this Hamiltonian system from Equations (4.6) and (4.7).

Proposition 4.3. The dispersion relation for this Hamiltonian system satisfies

$$\omega^4(k) - g\rho|k|\frac{1 + \coth(h_1|k|)}{\rho\coth(h_1|k|) + \rho_1}\omega^2(k) + g^2(\rho - \rho_1)|k|^2\frac{1}{\rho\coth(h_1|k|) + \rho_1} = 0. \tag{4.9}$$

Proof. In Fourier space, monochromatic plane waves have the form

$$\begin{cases} \widehat{\eta}(k) = \alpha(k)e^{i(kx-\omega(k)t)}; \quad \widehat{\xi}(k) = \beta(k)e^{i(kx-\omega(k)t)} \\ \widehat{\eta}_1(k) = \alpha_1(k)e^{i(kx-\omega(k)t)}; \quad \widehat{\xi}_1(k) = \beta_1(k)e^{i(kx-\omega(k)t)}. \end{cases} \tag{4.10}$$

Solving these four equations of motion,

$$\begin{aligned} (-i\omega(k)\alpha(k)) &= G_{11}^{(0)}B_0^{-1}G^{(0)}b(k) + G^{(0)}G_{12}^{(0)}B_0^{-1}\beta_1(k) \\ (-i\omega(k)\beta(k)) &= -g(\rho-\rho_1)\alpha(k) \\ (-i\omega(k)\alpha_1(k)) &= G^{(0)}G_{12}^{(0)}B_0^{-1}\beta(k) + \left(\rho_1^{-1}G_{22}^{(0)} - \rho\rho_1^{-1}B_0^{-1}(G_{12}^{(0)})^2\right)\beta_1(k) \\ (-i\omega(k)\beta_1(k)) &= -g\rho_1\alpha_1(k), \end{aligned} \tag{4.11}$$

we have

$$\alpha(k) = \frac{i\omega(k)\beta(k)}{g(\rho-\rho_1)}; \quad \alpha_1(k) = \frac{i\omega_1(k)\beta_1(k)}{g\rho_1} \tag{4.12}$$

$$\begin{pmatrix} \frac{\omega^2(k)}{g(\rho-\rho_1)}\beta(k) \\ \frac{\omega^2(k)}{g\rho_1}\beta_1(k) \end{pmatrix} = \begin{pmatrix} G_{11}^{(0)}B_0^{-1}G^{(0)} & G^{(0)}G_{12}^{(0)}B_0^{-1} \\ G^{(0)}G_{12}^{(0)}B_0^{-1} & \rho_1^{-1}G_{22}^{(0)} - \rho\rho_1^{-1}B_0^{-1}(G_{12}^{(0)})^2 \end{pmatrix} \begin{pmatrix} \beta(k) \\ \beta_1(k) \end{pmatrix}. \tag{4.13}$$

The dispersion relation is

$$0 = \det \begin{pmatrix} G_{11}^{(0)}B_0^{-1}G^{(0)} - \frac{\omega^2(k)}{g(\rho-\rho_1)} & G^{(0)}G_{12}^{(0)}B_0^{-1} \\ G^{(0)}G_{12}^{(0)}B_0^{-1} & \rho_1^{-1}G_{22}^{(0)} - \rho\rho_1^{-1}B_0^{-1}(G_{12}^{(0)})^2 - \frac{\omega^2(k)}{g\rho_1} \end{pmatrix}, \tag{4.14}$$

which after some algebraic manipulation can be simplified to Equation (4.9). □

Equation (4.9) has two branches of solutions

$$\omega^2(k) = \frac{g(1-\gamma)|k|\tanh(h_1|k|)}{1+\gamma\tanh(h_1|k|)}; \quad \omega_1^2(k) = g|k|, \tag{4.15}$$

where $\gamma := \frac{\rho_1}{\rho}$ is the density ratio between the upper and lower fluid domains. We associate the first root $\omega^2(k)$ with the wave motion of the interface while the second root $\omega_1^2(k)$ with the surface mode.

4.3 NORMAL MODE DECOMPOSITION

We perform a normal mode decomposition to diagonalise the quadratic part of the Hamiltonian, which will be expressed as

$$H^{(2)} = \frac{1}{2}\int_{\mathbb{R}} \left[\zeta\omega^2(D)\zeta + \mu^2 + \zeta_1\omega_1^2(D)\zeta_1 + \mu_1^2\right] dx. \tag{4.16}$$

This will involve two canonical transformations, the first of which is the rescaling

$$\begin{pmatrix} \eta' \\ \xi' \\ \eta_1' \\ \xi_1' \end{pmatrix} = \begin{pmatrix} \sqrt{g(\rho-\rho_1)} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{g(\rho-\rho_1)}} & 0 & 0 \\ 0 & 0 & \sqrt{g\rho_1} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{g\rho_1}} \end{pmatrix} \begin{pmatrix} \eta \\ \xi \\ \eta_1 \\ \xi_1 \end{pmatrix} =: \mathbf{M_1} \begin{pmatrix} \eta \\ \xi \\ \eta_1 \\ \xi_1 \end{pmatrix}. \tag{4.17}$$

After rescaling under this first canonical transformation and defining symbols $Q_a(D)$, $Q_b(D)$ and $Q_c(D)$, we rewrite the quadratic part of the kinetic energy

$$K^{(2)} = \frac{1}{2}\int_{\mathbb{R}} \left[\xi' Q_a(D)\xi' + 2\xi' Q_b(D)\xi_1' + \xi_1' Q_c(D)\xi_1'\right] dx. \tag{4.18}$$

Reading the quadratic parts of the Hamiltonian in Proposition 4.1 and rescaling by

$$\xi' = \frac{1}{\sqrt{g(\rho-\rho_1)}}\xi\,; \qquad \xi_1' = \frac{1}{\sqrt{g\rho_1}}\xi_1, \tag{4.19}$$

it follows that these symbols can be rewritten in terms of the Dirichlet-Neumann operators

$$\begin{cases} Q_a(D) &= g(\rho-\rho_1)G^{(0)}G_{11}^{(0)}B_0^{-1} \\ Q_b(D) &= -g\sqrt{\rho_1(\rho-\rho_1)}G^{(0)}G_{12}^{(0)}B_0^{-1} \\ Q_c(D) &= gG^{(0)}(\rho_1 G_{11}^{(0)} + \rho G^{(0)})B_0^{-1}, \end{cases} \tag{4.20}$$

where $B_0 = \rho G_{11}^{(0)} + \rho_1 G^{(0)}$. Using Equations (3.8), (3.23) and (3.24), these symbols are

$$\begin{cases} Q_a(D) &= \frac{g(\rho-\rho_1)|D|\coth(h_1 D)}{\rho\coth(h_1 D)+\rho_1 \mathrm{sgn}(D)} \\ Q_b(D) &= \frac{g\sqrt{\rho_1(\rho-\rho_1)}|D|\mathrm{csch}(h_1 D)}{\rho\coth(h_1 D)+\rho_1 \mathrm{sgn}(D)} \\ Q_c(D) &= \frac{g\rho_1|D|\coth(h_1 D)+g\rho D}{\rho\coth(h_1 D)+\rho_1 \mathrm{sgn}(D)}. \end{cases} \tag{4.21}$$

Moreover,

$$Q_a + Q_c = \frac{g\rho G^{(0)}(G_{11}^{(0)} + G^{(0)})}{B_0} \tag{4.22}$$

as well as

$$\sqrt{(Q_a - Q_c)^2 + 4Q_b^2} = \frac{g(2\rho_1 - \rho)(G^{(0)})^2 + g\rho G_{11}^{(0)} G^{(0)}}{B_0}. \tag{4.23}$$

The second canonical transformation is the rotation in phase space

$$\begin{pmatrix} \mu \\ \zeta \\ \mu_1 \\ \zeta_1 \end{pmatrix} = \begin{pmatrix} a^- & 0 & b^- & 0 \\ 0 & a^- & 0 & b^- \\ a^+ & 0 & b^+ & 0 \\ 0 & a^+ & 0 & b^+ \end{pmatrix} \begin{pmatrix} \eta' \\ \xi' \\ \eta_1' \\ \xi_1' \end{pmatrix} =: \mathbf{M_2} \begin{pmatrix} \eta' \\ \xi' \\ \eta_1' \\ \xi_1' \end{pmatrix}, \tag{4.24}$$

where the symbols are defined as

$$\begin{aligned} \theta(D) &= \frac{Q_c(D) - Q_a(D)}{Q_b(D)} \\ a^{\pm}(D) &= \left(2 + \frac{\theta^2}{2} \pm \frac{\theta}{2}\sqrt{4+\theta^2}\right)^{-1/2} \\ b^{\pm}(D) &= \frac{a^{\pm}(D)}{2}(\theta \pm \sqrt{4+\theta^2}) \end{aligned} \tag{4.25}$$

and a possible singularity in $\theta(D)$ is removed by the contributions from the numerator.

Next we take inverses of the composition of these two transformations to get

$$\begin{cases} \eta &= \frac{b^+}{\sqrt{g(\rho-\rho_1)}}\mu - \frac{b^-}{\sqrt{g(\rho-\rho_1)}}\mu_1 \\ \eta_1 &= -\frac{a^+}{\sqrt{g\rho_1}}\mu + \frac{a^-}{\sqrt{g\rho_1}}\mu_1. \end{cases} ; \begin{cases} \xi &= b^+\sqrt{g(\rho-\rho_1)}\zeta - b^-\sqrt{g(\rho-\rho_1)}\zeta_1 \\ \xi_1 &= -a^+\sqrt{g\rho_1}\zeta + a^-\sqrt{g\rho_1}\zeta_1. \end{cases} \tag{4.26}$$

The following lemma provides us with some key relations for our analysis.

Lemma 4.4. *The following conditions on $a^{\pm}(D)$ and $b^{\pm}(D)$ hold*

$$\begin{cases} a^+(D) = -b^-(D) \\ a^-(D) = b^+(D) = \frac{\rho_1}{\rho}\left(\frac{Q_a(D)+Q_c(D)}{Q_b(D)}\right)a^+(D) \\ (a^+(D))^2 + (a^-(D))^2 = (b^+(D))^2 + (b^-(D))^2 = 1 \\ a^+(D)b^+(D) + a^-(D)b^-(D) = 0. \end{cases} \tag{4.27}$$

Proof. We first verify that

$$b^-(a^+)^{-1} = \frac{-\sqrt{\theta - \sqrt{4+\theta^2}}}{\sqrt{4+\theta^2 - \theta\sqrt{4+\theta^2}}}\frac{\sqrt{4+\theta^2+\theta\sqrt{4+\theta}}}{2} = -1 \tag{4.28}$$

and similarly for $a^-(D) = b^+(D)$. Next we check that

$$(a^+)^2 + (a^-)^2 = \frac{2}{4+\theta^2+\theta\sqrt{4+\theta^2}} + \frac{2}{4+\theta^2-\theta\sqrt{4+\theta^2}} = 1 \tag{4.29}$$

as well as

$$(b^+)^2 + (b^-)^2 = \frac{(\theta - \sqrt{4+\theta^2})^2}{2(4+\theta^2-\theta\sqrt{4+\theta^2})} + \frac{(\theta + \sqrt{4+\theta^2})^2}{2(4+\theta^2+\theta\sqrt{4+\theta^2})} = 1, \tag{4.30}$$

and lastly

$$a^+b^+ + a^-b^- = \frac{\theta + \sqrt{4+\theta^2}}{4+\theta^2+\theta\sqrt{4+\theta^2}} + \frac{\theta - \sqrt{4+\theta^2}}{4+\theta^2-\theta\sqrt{4+\theta^2}} = 0, \tag{4.31}$$

which completes the derivation. □

We also note the even character or parity of the symbols $a^{\pm}$ and $b^{\pm}$.

Lemma 4.5. *The symbols of Q_a, Q_b, Q_c, $a^{\pm}$ and $b^{\pm}$ are all even.*

Proof. We first observe that $Q_a(D)$ has even character, since

$$Q_a(D) = \frac{g(\rho-\rho_1)|D|\coth(h_1D)}{\rho\coth(h_1D) + \rho_1\mathrm{sgn}(D)} = \frac{g(\rho-\rho_1)|D|}{\rho + \rho_1\tanh(h_1|D|)}. \tag{4.32}$$

Similarly, we calculate the remaining two symbols

$$Q_b(D) = \frac{g\sqrt{\rho_1(\rho-\rho_1)}|D|\mathrm{csch}(h_1D)}{\rho\coth(h_1D) + \rho_1\mathrm{sgn}(D)} = \frac{g\sqrt{\rho_1(\rho-\rho_1)}|D|\,\mathrm{csch}(h_1|D|)}{\rho\coth(h_1|D|) + \rho_1} \tag{4.33}$$

and

$$Q_c(D) = \frac{g\rho_1|D|\coth(h_1 D) + g\rho D}{\rho\coth(h_1 D) + \rho_1 \mathrm{sgn}(D)} = \frac{g\rho_1|D| + g\rho|D|\tanh(h_1|D|)}{\rho + \rho_1\tanh(h_1|D|)}. \quad (4.34)$$

Since Q_a, Q_b and Q_c are all even, so is $\theta = \frac{Q_c - Q_a}{Q_b}$ and thus so are $a^\pm$ and $b^\pm$ as these symbols are in turn implicitly defined by θ themselves. □

Before finding the quadratic terms of the Hamiltonian in normal coordinates, we find the eigenvalues and eigenvectors of the symmetric matrix

$$\begin{pmatrix} Q_a(D) & Q_b(D) \\ Q_b(D) & Q_c(D) \end{pmatrix}. \quad (4.35)$$

The eigenvalues are the roots of characteristic equation

$$\left| \begin{pmatrix} Q_a & Q_b \\ Q_b & Q_c \end{pmatrix} - \lambda \mathbf{I} \right| = \lambda^2 - (Q_a + Q_c)\lambda + (Q_a Q_c - Q_b^2), \quad (4.36)$$

which agree with the solutions of the dispersion relation

$$\omega^2(D) = \frac{g(1-\gamma)|D|\tanh(h_1|D|)}{1 + \gamma\tanh(h_1|D|)}; \quad \omega_1^2(D) = g|D|. \quad (4.37)$$

Proposition 4.6. The eigenvalues of the symmetric matrix

$$\begin{pmatrix} Q_a(D) & Q_b(D) \\ Q_b(D) & Q_c(D) \end{pmatrix}. \quad (4.38)$$

given by $\lambda \leqslant \lambda_1$ are the internal and surface modes, $\omega^2(D)$ and $\omega_1^2(D)$, respectively.

Proof. We directly calculate

$$\begin{aligned} \lambda(D) &= \frac{1}{2}\left(Q_a(D) + Q_c(D) - \sqrt{(Q_a(D) - Q_c(D))^2 + 4Q_b^2(D)} \right) \\ &= \frac{1}{2}\frac{g|D|(1 + \tanh(h_1|D|))}{2(1 + \gamma\tanh(h_1|D|))} - \frac{g|D|\left(1 - (1-2\gamma)\tanh(h_1|D|)\right)}{2(1+\gamma\tanh(h_1|D|))} \\ &= \frac{g(1-\gamma)|D|\tanh(h_1|D|)}{1 + \gamma\tanh(h_1|D|)} \end{aligned} \quad (4.39)$$

and

$$\begin{aligned}\lambda_1(D) &= \frac{1}{2}\left(Q_a(D) + Q_c(D) + \sqrt{(Q_a(D) - Q_c(D))^2 + 4Q_b^2(D)}\right) \\ &= \frac{g|D|(1 + \tanh(h_1|D|))}{2(1 + \gamma\tanh(h_1|D|))} + \frac{g|D|\,(1 - (1 - 2\gamma)\tanh(h_1|D|))}{2(1 + \gamma\tanh(h_1|D|))} \\ &= g|D| \end{aligned} \tag{4.40}$$

to complete the result. □

Moreover, we show the normalised eigenvectors are $(a^-, b^-)^T$ and $(a^+, b^+)^T$, respectively.

Lemma 4.7. *The eigenvectors of the symmetric matrix*

$$\begin{pmatrix} Q_a(D) & Q_b(D) \\ Q_b(D) & Q_c(D) \end{pmatrix} \tag{4.41}$$

that correspond to internal and surface modes, $\omega^2(D)$ and $\omega_1^2(D)$, are $(a^-, b^-)^T$ and $(a^+, b^+)^T$, respectively.

Proof. First we show that $(a^-, a^+)^T$ lies in the kernel space of

$$\begin{pmatrix} Q_a(D) - \omega^2(D) & Q_b(D) \\ Q_b(D) & Q_c(D) - \omega^2(D) \end{pmatrix}, \tag{4.42}$$

which follows as

$$\begin{aligned} & a^-(Q_a - \omega^2) + b^- Q_b \\ &\quad = \frac{\rho_1}{\rho}\frac{Q_a + Q_c}{Q_b} a^+ \left(\frac{1}{2}(Q_a - Q_c) + \frac{1}{2}\sqrt{(Q_a - Q_c)^2 + 4Q_b^2}\right) - a^+ B \\ &\quad = \frac{a^+}{B}\left(\frac{g\rho_1 G^{(0)}(G_{11}^{(0)} + G^{(0)})}{B_0}(gG^{(0)} - Q_c(D)) - Q_b^2(D)\right) \\ &\quad = 0. \end{aligned} \tag{4.43}$$

and

$$\begin{aligned} a^- Q_b + b^-(Q_c - \omega^2) &= \frac{\rho_1}{\rho}\frac{Q_a + Q_c}{Q_b} a^+ Q_b - a^+(\omega_1^2 - Q_a) \\ &= a^+\left(\frac{g\rho_1 G^{(0)}(G_{11}^{(0)} + G^{(0)})}{B_0} + (Q_a(D) - gG^{(0)})\right) \\ &= 0. \end{aligned} \tag{4.44}$$

Since the matrix is symmetric, the eigenvectors are orthogonal and it follows that $(a^+, b^+)^T$ is the remaining eigenvector for the surface mode, which completes the proof. □

Through these two transformations, we also write the equations of motion in normal coordinates. These variables are also canonical since the symplectic matrix remains unchanged.

Lemma 4.8. *Hamilton's equations for transformed variables $(\mu, \zeta, \mu_1, \zeta_1)$ is*

$$\partial_t \begin{pmatrix} \mu \\ \zeta \\ \mu_1 \\ \zeta_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_\mu H \\ \delta_\zeta H \\ \delta_{\mu_1} H \\ \delta_{\zeta_1} H \end{pmatrix} =: \mathbf{J} \begin{pmatrix} \delta_\mu H \\ \delta_\zeta H \\ \delta_{\mu_1} H \\ \delta_{\zeta_1} H \end{pmatrix}. \tag{4.45}$$

Proof. Since the change of variables matrix $\mathbf{M_2 M_1}$ is orthogonal and commutes with $\mathbf{J}$, it follows

$$\partial_t \begin{pmatrix} \mu \\ \zeta \\ \mu_1 \\ \zeta_1 \end{pmatrix} = (\mathbf{M_2 M_1}) \partial_t \begin{pmatrix} \eta \\ \xi \\ \eta_1 \\ \xi_1 \end{pmatrix} = (\mathbf{M_2 M_1}) \mathbf{J} (\mathbf{M_2 M_1})^T \begin{pmatrix} \delta_\mu H \\ \delta_\zeta H \\ \delta_{\mu_1} H \\ \delta_{\zeta_1} H \end{pmatrix} = \mathbf{J} \begin{pmatrix} \delta_\mu H \\ \delta_\zeta H \\ \delta_{\mu_1} H \\ \delta_{\zeta_1} H \end{pmatrix}, \tag{4.46}$$

which completes the proof. □

Proposition 4.9. The quadratic part of the Hamiltonian is

$$H^{(2)} = \frac{1}{2} \int_{\mathbb{R}} \left[\zeta \omega^2(D) \zeta + \mu^2 + \zeta_1 \omega_1^2(D) \zeta_1 + \mu_1^2 \right] dx. \tag{4.47}$$

in normal variables.

Proof. We calculate the integrand of the potential energy

$$\begin{aligned} \frac{1}{2} g(\rho - \rho_1)\eta^2 + \frac{1}{2} g \rho_1 \eta_1^2 &= \frac{1}{2} \eta'^2 + \frac{1}{2} \eta_1'^2 \\ &= \frac{1}{2} (a_- \eta' + b_- \eta_1')^2 + \frac{1}{2} (a_+ \eta' + b_+ \eta_1')^2 \\ &= \frac{1}{2} \mu^2 + \frac{1}{2} \mu_1^2 \end{aligned} \tag{4.48}$$

as well as the integrand of the kinetic energy

$$\begin{aligned}
&\frac{1}{2}\begin{pmatrix}\xi' & \xi_1'\end{pmatrix}\begin{pmatrix}Q_a(D) & Q_b(D)\\ Q_b(D) & Q_c(D)\end{pmatrix}\begin{pmatrix}\xi'\\ \xi_1'\end{pmatrix}\\
&\quad=\frac{1}{2}\begin{pmatrix}\zeta & \zeta_1\end{pmatrix}\begin{pmatrix}b^+ & -a^+\\ -b^- & a^-\end{pmatrix}\begin{pmatrix}Q_a(D) & Q_b(D)\\ Q_b(D) & Q_c(D)\end{pmatrix}\begin{pmatrix}b^+ & -b^-\\ -a^+ & a^-\end{pmatrix}\begin{pmatrix}\zeta\\ \zeta_1\end{pmatrix}\\
&\quad=\frac{1}{2}\begin{pmatrix}\zeta & \zeta_1\end{pmatrix}\begin{pmatrix}\omega^2(D) & 0\\ 0 & \omega_1^2(D)\end{pmatrix}\begin{pmatrix}\zeta\\ \zeta_1\end{pmatrix}\\
&\quad=\frac{1}{2}\zeta\omega^2(D)\zeta+\frac{1}{2}\zeta_1\omega_1^2(D)\zeta_1,
\end{aligned} \tag{4.49}$$

where $\omega^2(D)$ and $\omega_1^2(D)$ are the internal and surface modes. From Equations (4.3) and (4.18), we combine Equations (4.48) and (4.49) to get the result. □

4.4 THE MOMENTUM

The impulse integral, or momentum, can also be written in both canonical and normal coordinates. Since the Poisson commutator $\{H, I\} = 0$, it follows that the momentum is an invariant of motion that does not affect our analysis in the Hamiltonian setting.

Proposition 4.10 ([CGS20])**.** In terms of the canonical coordinates, the momentum I has the form

$$I := \int_{\mathbb{R}}\left(\rho\int_{-\infty}^{\eta}\partial_x\phi\,dy+\rho_1\int_{\eta}^{h_1+\eta_1}\partial_x\phi_1\,dy\right)dx = -\int_{\mathbb{R}}(\xi\partial_x\eta+\xi_1\partial_x\eta_1)\,dx, \tag{4.50}$$

while, in normal coordinates, the momentum is

$$I = -\int_{\mathbb{R}}(\zeta\partial_x\mu+\zeta_1\partial_x\mu_1)\,dx. \tag{4.51}$$

Proof. In canonical coordinates,

$$
\begin{aligned}
I &= \int_{\mathbb{R}} \left(\rho \int_{-\infty}^{\eta} \partial_x \phi \, dy + \rho_1 \int_{\eta}^{h_1+\eta_1} \partial_x \phi_1 \, dy \right) dx \\
&= \int_{\mathbb{R}} \rho \left(\partial_x \left(\int_{-\infty}^{\eta} \phi \, dy \right) - \phi(x,\eta,t)(\partial_x \eta) \right) \\
&\quad + \rho_1 \left(\partial_x \left(\int_{\eta}^{h_1+\eta_1} \phi_1 \, dy \right) - \phi_1(x,\eta_1,t)(\partial_x \eta_1) + \phi_1(x,\eta,t)(\partial_x \eta) \right) dx \\
&= \rho \int_{-\infty}^{\eta} \phi \, dy \,\Big|_{x=-\infty}^{\infty} + \rho_1 \int_{\eta}^{h_1+\eta_1} \phi_1 \, dy \,\Big|_{x=-\infty}^{\infty} \\
&\quad - \int_{\mathbb{R}} \Big[(\rho\Phi - \rho_1 \Phi_1)(\partial_x \eta) + \rho_1 \Phi_2 (\partial_x \eta_2) \Big] dx \\
&= - \int_{\mathbb{R}} (\xi \partial_x \eta + \xi_1 \partial_x \eta_1) \, dx,
\end{aligned} \tag{4.52}
$$

where we recall $\xi = \rho\Phi - \rho_1\Phi_1$ and $\xi_1 = \rho_1\Phi_2$. Next, using the canonical transformation and Lemma 4.4, we express

$$
\begin{aligned}
I &= - \int_{\mathbb{R}} (\xi \partial_x \eta + \xi_1 \partial_x \eta_1) \, dx \\
&= - \int_{\mathbb{R}} \big[\left(b^+ \zeta - b^- \zeta_1 \right) \left(b^+ \partial_x \mu - b^- \partial_x \mu_1 \right) \\
&\quad + \left(a^+ \zeta - a^- \zeta_1 \right) \left(a^+ \partial_x \mu - a^- \partial_x \mu_1 \right) \big] \, dx \\
&= - \int_{\mathbb{R}} [((b^+)^2 + (a^+)^2) \zeta \partial_x \mu - (a^+ b^+ + a^- b^-) \zeta \partial_x \mu_1 \\
&\quad - (a^+ b^+ + a^- b^-) \zeta_1 \partial_x \mu + ((b^+)^2 + (a^+)^2) \zeta \partial_x \mu] \, dx \\
&= - \int_{\mathbb{R}} (\zeta \partial_x \mu + \zeta_1 \partial_x \mu_1) \, dx,
\end{aligned} \tag{4.53}
$$

which yields the desired result. □

5

CUBIC TERMS OF THE HAMILTONIAN

5.1 CUBIC TERMS OF THE HAMILTONIAN IN VARIABLES $(\eta, \eta_1, \xi, \xi_1)$

We compute the cubic parts of the Hamiltonian in terms of canonical variables $(\eta, \eta_1, \xi, \xi_1)$, while we will transform to normal variables $(\zeta, \zeta_1, \mu, \mu_1)$ in later sections. As the potential energy V is quadratic and does not contribute any higher-order terms, the cubic part of the Hamiltonian is simply the cubic part of the kinetic energy. Recalling that the kinetic energy K

$$\begin{aligned} K &= \frac{1}{2}\int_{\mathbb{R}} \xi G_{11} B^{-1} G(\eta)\xi - \xi G(\eta) B^{-1} G_{12}\xi_1 - \xi_1 G_{21} B^{-1} G(\eta)\xi \\ &\quad + \xi_1 \left(\rho_1^{-1} G_{22} - \rho\rho_1^{-1} G_{21} B^{-1} G_{12}\right) \xi_1 \, dx \\ &= \frac{1}{2}\int_{\mathbb{R}} \xi G_{11} B^{-1} G(\eta)\xi - 2\xi G(\eta) B^{-1} G_{12}\xi_1 \\ &\quad + \xi_1 \left(\rho_1^{-1} G_{22} - \rho\rho_1^{-1} G_{21} B^{-1} G_{12}\right) \xi_1 \, dx \\ &=: (\mathrm{I}) - (\mathrm{II}) + (\mathrm{III}), \end{aligned} \tag{5.1}$$

where

$$\begin{cases} (\mathrm{I}) = \frac{1}{2}\int_{\mathbb{R}} \xi G_{11} B^{-1} G(\eta)\xi \, dx \\ (\mathrm{II}) = \int_{\mathbb{R}} \xi G(\eta) B^{-1} G_{12}\xi_1 \, dx \\ (\mathrm{III}) = \frac{1}{2}\int_{\mathbb{R}} \xi_1 \left(\rho_1^{-1} G_{22} - \rho\rho_1^{-1} G_{21} B^{-1} G_{12}\right) \xi_1 \, dx, \end{cases} \tag{5.2}$$

identify the first, second and third terms of Equation (5.1), respectively.

Proposition 5.1. The cubic part of (I) is

$$\begin{aligned} (\mathrm{I})^{(3)} = \frac{1}{2}\int_{\mathbb{R}} \Big[&-\rho\eta \, (D B_0^{-1} G_{11}^{(0)} \xi)^2 - (\rho - \rho_1)\eta \, (G^{(0)} B_0^{-1} G_{11}^{(0)} \xi)^2 \\ &+ \rho_1 \eta \, (D B_0^{-1} G^{(0)} \xi)^2 - \rho_1 \eta_1 \, (G_{12}^{(0)} B_0^{-1} G^{(0)} \xi)^2 \Big] \, dx. \end{aligned} \tag{5.3}$$

Proof. To calculate term (I), using the expansions of Propositions 3.1 and 3.2 from Chapter 2, we first expand the integrand in powers of η and η_1 at order $\mathcal{O}(\eta, \eta_1)$

$$\begin{aligned}
G_{11}B^{-1}G(\eta) &= (G_{11}^{(0)} + G_{11}^{(10)} + G_{11}^{(01)}) \left(\frac{1}{B_0} - \frac{1}{B_0} B^{(1)} \frac{1}{B_0} \right) (G^{(0)} + G^{(1)}) \\
&\quad + \mathcal{O}(|(\eta, \eta_1)|^2) \\
&= G_{11}^{(0)} B_0^{-1} G^{(0)} + (G_{11}^{(10)} + G_{11}^{(01)}) B_0^{-1} G^{(0)} + G_{11}^{(0)} B_0^{-1} G^{(1)} \\
&\quad - G_{11}^{(0)} B_0^{-1} (\rho G_{11}^{(10)} + \rho G_{11}^{(01)} + \rho_1 G^{(1)}) B_0^{-1} G^{(0)} + \mathcal{O}(|(\eta, \eta_1)|^2) \\
&= G_{11}^{(0)} B_0^{-1} G^{(0)} + \left(1 - \rho G_{11}^{(0)} B_0^{-1}\right) (G_{11}^{(10)} + G_{11}^{(01)}) G^{(0)} B_0^{-1} \\
&\quad + G_{11}^{(0)} B_0^{-1} G^{(1)} \left(1 - \rho_1 G^{(0)} B_0^{-1}\right) + \mathcal{O}(|(\eta, \eta_1)|^2) \\
&= \frac{G_{11}^{(0)} G^{(0)}}{B_0} + \frac{\rho_1 G^{(0)}}{B_0} (G_{11}^{(10)} + G_{11}^{(01)}) \frac{G^{(0)}}{B_0} \\
&\quad + \frac{\rho G_{11}^{(0)}}{B_0} G^{(1)} \frac{G_{11}^{(0)}}{B_0} + \mathcal{O}(|(\eta, \eta_1)|^2).
\end{aligned} \tag{5.4}$$

We can simplify the terms

$$\begin{aligned}
\rho_1 \frac{G^{(0)}}{B_0} G_{11}^{(10)} \frac{G^{(0)}}{B_0} &= \rho_1 \frac{G^{(0)}}{B_0} (G_{11}^{(0)} \eta G_{11}^{(0)} - D\eta D) \frac{G^{(0)}}{B_0} \\
&= \rho_1 G^{(0)} B_0^{-1} G_{11}^{(0)} \eta G^{(0)} G_{11}^{(0)} B_0^{-1} - \rho_1 \frac{G^{(0)} D}{B_0} \eta \frac{G^{(0)} D}{B_0},
\end{aligned} \tag{5.5}$$

as well as

$$\begin{aligned}
\rho_1 \frac{G^{(0)}}{B_0} G_{11}^{(01)} \frac{G^{(0)}}{B_0} &= \rho_1 \frac{G^{(0)}}{B_0} (-G_{12}^{(0)} \eta_1 G_{12}^{(0)}) \frac{G^{(0)}}{B_0} \\
&= -\rho_1 \frac{G^{(0)} G_{12}^{(0)}}{\rho G_{11}^{(0)} + \rho_1 G^{(0)}} \eta_1 \frac{G^{(0)} G_{12}^{(0)}}{\rho G_{11}^{(0)} + \rho_1 G^{(0)}}
\end{aligned} \tag{5.6}$$

and finally

$$\begin{aligned}
&\rho \frac{G_{11}^{(0)}}{B_0} G^{(1)} \frac{G_{11}^{(0)}}{B_0} \\
&= \rho \frac{G_{11}^{(0)}}{B_0} (D\eta D - G^{(0)} \eta G^{(0)}) \frac{G_{11}^{(0)}}{B_0} \\
&= \rho G_{11}^{(0)} D B_0^{-1} \eta G_{11}^{(0)} D B_0^{-1} - \rho G_{11}^{(0)} G^{(0)} B_0^{-1} \eta G_{11}^{(0)} G^{(0)} B_0^{-1}.
\end{aligned} \tag{5.7}$$

We can then use the self-adjoint character of the operators to simplify

$$
\begin{aligned}
\text{(I)} &= \frac{1}{2}\int \xi G_{11} B^{-1} G(\eta)\xi \, dx \\
&= \frac{1}{2}\int_{\mathbb{R}} \Big[\xi G_{11}^{(0)} B_0^{-1} G^{(0)}\xi + (\rho_1 - \rho)\xi G_{11}^{(0)} G^{(0)} B_0^{-1}\eta G_{11}^{(0)} G^{(0)} B_0^{-1}\xi \\
&\quad - \rho_1 \xi G^{(0)} D B_0^{-1}\eta G^{(0)} D B_0^{-1}\xi + \rho\xi G_{11}^{(0)} D B_0^{-1}\eta G_{11}^{(0)} D B_0^{-1}\xi \\
&\quad - \rho_1 \xi G_{12}^{(0)} G^{(0)} B_0^{-1}\eta_1 G_{12}^{(0)} G^{(0)} B_0^{-1}\xi\Big] \, dx \\
&= \frac{1}{2}\int_{\mathbb{R}} \xi G_{11}^{(0)} G^{(0)} B_0^{-1}\xi \, dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}} \Big[-\rho\eta\ (D B_0^{-1} G_{11}^{(0)}\xi)^2 - (\rho - \rho_1)\eta\ (G^{(0)} B_0^{-1} G_{11}^{(0)}\xi)^2 \\
&\quad + \rho_1\eta\ (D B_0^{-1} G^{(0)}\xi)^2 - \rho_1\eta_1\ (G_{12}^{(0)} B_0^{-1} G^{(0)}\xi)^2\Big] \, dx + \mathcal{O}(|(\eta,\eta_1)|^2), \qquad (5.8)
\end{aligned}
$$

which completes the derivation. □

We move on to the second term (II) in our analysis.

Proposition 5.2. The cubic part of (II) is

$$
\begin{aligned}
\text{(II)}^{(3)} = \int_{\mathbb{R}} &\Big[\rho\eta (D B_0^{-1} G_{11}^{(0)}\xi)(D B_0^{-1} G_{12}^{(0)}\xi_1) \\
&+ (\rho - \rho_1)\eta (G^{(0)} B_0^{-1} G_{11}^{(0)}\xi)(G^{(0)} B_0^{-1} G_{12}^{(0)}\xi_1) + \rho\eta (D G^{(0)} B_0^{-1}\xi)(D B_0^{-1} G_{12}^{(0)}\xi_1) \\
&+ \eta_1 (G_{12}^{(0)} G^{(0)} B_0^{-1}\xi)(G_{11}^{(0)}\xi_1 - \rho (G_{12}^{(0)})^2 B_0^{-1}\xi_1)\Big] \, dx. \qquad (5.9)
\end{aligned}
$$

Proof. We expand the integrand $G(\eta) B^{-1} G_{12}$ in powers of (η, η_1)

$$
\begin{aligned}
&G(\eta) B^{-1} G_{12} \\
&= (G^{(0)} + G^{(1)}) \left(\frac{1}{B_0} - \frac{1}{B_0} B^{(1)} \frac{1}{B_0}\right) (G_{12}^{(0)} + G_{12}^{(10)} + G_{12}^{(01)}) + \mathcal{O}(|(\eta,\eta_1)|^2) \\
&= \frac{G^{(0)} G_{12}^{(0)}}{B_0} + G^{(1)} \frac{1}{B_0} G_{12}^{(0)} - G^{(0)} \frac{1}{B_0} B^{(1)} \frac{1}{B_0} G_{12}^{(0)} \\
&\quad + \frac{G^{(0)}}{B_0} (G_{12}^{(10)} + G_{12}^{(01)}) + \mathcal{O}(|(\eta,\eta_1)|^2) \\
&= G^{(0)} G_{12}^{(0)} B_0^{-1} + (D\eta D - G^{(0)}\eta G^{(0)}) B_0^{-1} G_{12}^{(0)} + G^{(0)} B_0^{-1} G_{11}^{(0)}\eta G_{12}^{(0)} \\
&\quad - G^{(0)} B_0^{-1} G_{12}^{(0)}\eta_1 G_{11}^{(0)} - \rho G^{(0)} B_0^{-1} (G_{11}^{(0)}\eta G_{11}^{(0)} - D\eta D) B_0^{-1} G_{12}^{(0)} \\
&\quad + \rho G^{(0)} B_0^{-1} G_{12}^{(0)}\eta_1 G_{12}^{(0)} B_0^{-1} G_{12}^{(0)} - \rho_1 G^{(0)} B_0^{-1} D\eta D B_0^{-1} G_{12}^{(0)} \\
&\quad + \rho_1 G^{(0)} B_0^{-1} G^{(0)}\eta G^{(0)} B_0^{-1} G_{12}^{(0)} + \mathcal{O}(|(\eta,\eta_1)|^2). \qquad (5.10)
\end{aligned}
$$

Next we group the third and fourth terms

$$\begin{aligned} & G^{(0)} G_{11}^{(0)} B_0^{-1} \eta G_{12}^{(0)} - G^{(0)} G_{11}^{(0)} B_0^{-1} \eta G_{12}^{(0)} \rho G_{11}^{(0)} B_0^{-1} \\ & \quad = \rho_1 G^{(0)} G_{11}^{(0)} B_0^{-1} \eta G^{(0)} G_{12}^{(0)} B_0^{-1}, \end{aligned} \tag{5.11}$$

the second and seventh η terms

$$\begin{aligned} & - G^{(0)} \eta G^{(0)} G_{12}^{(0)} B_0^{-1} + \rho_1 (G^{(0)})^2 B_0^{-1} \eta G^{(0)} G_{12}^{(0)} B_0^{-1} \\ & \quad = -\rho G_{11}^{(0)} G^{(0)} B_0^{-1} \eta G^{(0)} G_{12}^{(0)} B_0^{-1}, \end{aligned} \tag{5.12}$$

as well as the first and sixth η terms

$$D\eta D G_{12}^{(0)} B_0^{-1} - \rho_1 D G^{(0)} B_0^{-1} \eta D G_{12}^{(0)} B_0^{-1} = \rho D G_{11}^{(0)} B_0^{-1} \eta D G_{12}^{(0)} B_0^{-1}, \tag{5.13}$$

in which the variable η appears. Using these identities, we group all of the terms that include η together

$$\begin{aligned} & (\rho_1 - \rho) G^{(0)} G_{11}^{(0)} B_0^{-1} \eta G^{(0)} G_{12}^{(0)} B_0^{-1} + \rho G_{11}^{(0)} D B_0^{-1} \eta D G_{12}^{(0)} B_0^{-1} \\ & \quad + \rho G^{(0)} D B_0^{-1} \eta G_{12}^{(0)} D B_0^{-1}. \end{aligned} \tag{5.14}$$

Similarly, using the relation $(G_{11}^{(0)})^2 - (G_{12}^{(0)})^2 = (G^{(0)})^2$, we calculate the terms in (II) that include the variable η_1

$$\begin{aligned} & - G^{(0)} G_{12}^{(0)} B_0^{-1} \eta_1 G_{11}^{(0)} + \rho G^{(0)} G_{12}^{(0)} B_0^{-1} \eta_1 (G_{12}^{(0)})^2 B_0^{-1} \\ & \quad = -\rho_1 G^{(0)} G_{12}^{(0)} B_0^{-1} \eta_1 G_{11}^{(0)} G^{(0)} B_0^{-1} - \rho G^{(0)} G_{12}^{(0)} B_0^{-1} \eta_1 (G_{12}^{(0)})^2 B_0^{-1}. \end{aligned} \tag{5.15}$$

We can then use the self-adjoint character of the operators to simplify

$$\begin{aligned} (\mathrm{II}) = & \int_{\mathbb{R}} \xi \Big[G^{(0)} B_0^{-1} G_{12}^{(0)} + G_{11}^{(0)} D B_0^{-1} \eta D G_{12}^{(0)} B_0^{-1} \\ & + (\rho_1 - \rho) G^{(0)} G_{11}^{(0)} B_0^{-1} \eta G^{(0)} G_{12}^{(0)} B_0^{-1} + \rho G^{(0)} D B_0^{-1} \eta G_{12}^{(0)} D B_0^{-1} \\ & + \rho G^{(0)} G_{12}^{(0)} B_0^{-1} \eta_1 (G_{12}^{(0)})^2 B_0^{-1} - G^{(0)} G_{12}^{(0)} B_0^{-1} \eta_1 G_{11}^{(0)} \Big] \xi_1 \, dx \\ = & \int_{\mathbb{R}} \xi G^{(0)} G_{12}^{(0)} B_0^{-1} \xi_1 \, dx + \int_{\mathbb{R}} \Big[\rho \eta (D B_0^{-1} G_{11}^{(0)} \xi)(D B_0^{-1} G_{12}^{(0)} \xi_1) \\ & + (\rho - \rho_1) \eta (G^{(0)} B_0^{-1} G_{11}^{(0)} \xi)(G^{(0)} B_0^{-1} G_{12}^{(0)} \xi_1) \\ & + \rho \eta (D G^{(0)} B_0^{-1} \xi)(D B_0^{-1} G_{12}^{(0)} \xi_1) \\ & + \eta_1 (G_{12}^{(0)} G^{(0)} B_0^{-1} \xi)(G_{11}^{(0)} \xi_1 - \rho (G_{12}^{(0)})^2 B_0^{-1} \xi_1) \Big] \, dx \\ & + \mathcal{O}(|(\eta, \eta_1)|^2), \end{aligned} \tag{5.16}$$

which completes our proof. □

Lastly we turn to the third and final term (III).

Proposition 5.3. The cubic part of (III) is

$$\begin{aligned}(\mathrm{III})^{(3)} = \frac{1}{2}\int \Big[&(\rho_1-\rho)\eta(G^{(0)}B_0^{-1}G_{12}^{(0)}\xi)^2 - \frac{\eta_1}{\rho_1}(G_{22}^{(0)}\xi_1 - \rho(G_{12}^{(0)})^2B_0^{-1}\xi_1)^2 \\ &- \frac{1}{\rho_1}\eta_1(D\xi_1)^2 + \frac{\rho}{\rho_1}(\rho-\rho_1)\eta(DB_0^{-1}G_{12}^{(0)}\xi_1)^2\Big]\,dx. \qquad (5.17)\end{aligned}$$

Proof. We expand the integrand $\rho_1^{-1}G_{22} - \rho\rho_1^{-1}G_{21}B^{-1}G_{12}$

$$\begin{aligned}&\rho_1^{-1}G_{22} - \rho\rho_1^{-1}G_{21}B^{-1}G_{12} \\ &\quad= \frac{1}{\rho_1}(G_{22}^{(0)} + G_{22}^{(10)} + G_{22}^{(01)}) \\ &\qquad - \frac{\rho}{\rho_1}(G_{21}^{(0)} + G_{21}^{(10)} + G_{21}^{(01)})\left(\frac{1}{B_0} - \frac{1}{B_0}B^{(1)}\frac{1}{B_0}\right)(G_{12}^{(0)} + G_{12}^{(10)} + G_{12}^{(01)}) \\ &\quad= \frac{1}{\rho_1}G_{11}^{(0)} - \frac{\rho}{\rho_1}G_{12}^{(0)}B_0^{-1}G_{12}^{(0)} + \frac{1}{\rho_1}(G_{22}^{(10)} + G_{22}^{(01)}) \\ &\qquad - \frac{\rho}{\rho_1}(G_{21}^{(10)} + G_{21}^{(01)})B_0^{-1}G_{12}^{(0)} - \frac{\rho}{\rho_1}G_{12}^{(0)}B_0^{-1}(G_{12}^{(10)} + G_{12}^{(01)}) \\ &\qquad + \frac{\rho}{\rho_1}G_{12}^{(0)}B_0^{-1}(\rho(G_{11}^{(10)} + G_{11}^{(01)}) + \rho_1 G^{(1)})B_0^{-1}G_{12}^{(0)} + \mathcal{O}(|(\eta,\eta_1)|^2).\end{aligned} \qquad (5.18)$$

We proceed by simplifying the terms that include the variable η to

$$\begin{aligned}&G_{12}^{(0)}\eta G_{12}^{(0)}G^{(0)}B_0^{-1} - \rho G_{12}^{(0)}G_{11}^{(0)}B_0^{-1}\eta G^{(0)}G_{12}^{(0)}B_0^{-1} \\ &\qquad + \frac{\rho(\rho_1-\rho)}{\rho_1}G_{12}^{(0)}DB_0^{-1}\eta G_q 12^{(0)}DB_0^{-1} - \rho G_{12}^{(0)}G^{(0)}B_0^{-1}\eta G^{(0)}G_{12}^{(0)}B_0^{-1} \\ &\quad= (\rho_1-\rho)G_{12}^{(0)}G^{(0)}B_0^{-1}\eta G^{(0)}G_{12}^{(0)}B_0^{-1} + \frac{\rho}{\rho_1}G_{12}^{(0)}DB_0^{-1}\eta G_{12}^{(0)}DB_0^{-1} \qquad (5.19)\end{aligned}$$

as well as those including η_1

$$\frac{1}{\rho_1}D\eta_1 D - \frac{1}{\rho_1}(G_{11}^{(0)} - \rho(G_{12}^{(0)})^2B_0^{-1})\eta_1(G_{11}^{(0)} - \rho(G_{12}^{(0)})^2B_0^{-1}). \qquad (5.20)$$

Again, one uses the self-adjoint character of the operators to get

$$
\begin{aligned}
(\mathrm{III}) &= \frac{1}{2}\int_{\mathbb{R}} \xi_1 \left(\rho_1^{-1} G_{22} - \rho\rho_1^{-1} G_{21} B^{-1} G_{12}\right) \xi_1 \, dx \\
&= \frac{1}{2}\int_{\mathbb{R}} \xi_1 \left(\rho_1^{-1} G_{22}^{(0)} - \rho\rho_1^{-1} B_0^{-1} (G_{12}^{(0)})^2\right) \xi_1 \, dx \\
&\quad + \frac{1}{2}\int \Big[(\rho_1 - \rho)\eta (G^{(0)} B_0^{-1} G_{12}^{(0)} \xi)^2 - \frac{\eta_1}{\rho_1} (G_{22}^{(0)} \xi_1 - \rho (G_{12}^{(0)})^2 B_0^{-1} \xi_1)^2 \\
&\quad - \frac{1}{\rho_1}\eta_1 (D\xi_1)^2 + \frac{\rho}{\rho_1}(\rho - \rho_1)\eta (D B_0^{-1} G_{12}^{(0)} \xi_1)^2 \Big] \, dx \\
&\quad + \mathcal{O}(|(\eta, \eta_1)|^2),
\end{aligned}
\tag{5.21}
$$

which completes the result. □

Adding the cubic parts of the three contributions, respectively, we find the third-order terms of the Hamiltonian.

Corollary 5.4. *In canonical variables, the cubic part of the Hamiltonian is*

$$
\begin{aligned}
H^{(3)} &= (\mathrm{I})^{(3)} - (\mathrm{II})^{(3)} + (\mathrm{III})^{(3)} \\
&= \frac{1}{2}\int_{\mathbb{R}} \Big[- (\rho - \rho_1)\eta \left(G^{(0)} B_0^{-1} (G_{11}^{(0)} \xi - G_{12}^{(0)} \xi_1) \right)^2 \\
&\quad - \rho_1 \eta_1 \left(G_{12}^{(0)} G^{(0)} B_0^{-1} \xi - \frac{1}{\rho_1} (G_{11}^{(0)} \xi_1 - \rho (G_{12}^{(0)})^2 B_0^{-1} \xi_1) \right)^2 \\
&\quad - \rho\eta \left(D B_0^{-1} (G_{11}^{(0)} \xi - G_{12}^{(0)} \xi_1) \right)^2 \\
&\quad + \rho_1 \eta \left(D B_0^{-1} G^{(0)} \xi + \frac{\rho}{\rho_1} D G_{12}^{(0)} B_0^{-1} \xi_1 \right)^2 - \frac{1}{\rho_1}\eta_1 (D\xi_1)^2 \Big] \, dx \\
&=: R_1 + R_2 + R_3 + R_4 + R_5,
\end{aligned}
\tag{5.22}
$$

where we identify each term on the right-hand side to R_1, R_2, R_3, R_4 and R_5.

Proof. These identities follow directly from Propositions 5.1, 5.2 and 5.3. □

5.2 CUBIC TERMS OF THE HAMILTONIAN IN VARIABLES $(\mu, \mu_1, \zeta, \zeta_1)$

Having derived the cubic terms of the Hamiltonian from canonical variables $(\eta, \eta_1, \xi, \xi_1)$, we now convert it into normal variables $(\zeta, \zeta_1, \mu, \mu_1)$, by examining R_j for each $j \in \{1, 2, 3, 4, 5\}$ separately.

In R_1, from Equation (5.22), we rewrite the factor $G^{(0)}B_0^{-1}(G_{11}^{(0)}\xi - G_{12}^{(0)}\xi_1)$ as

$$\begin{aligned}
&G^{(0)}B_0^{-1}(G_{11}^{(0)}\xi - G_{12}^{(0)}\xi_1)\\
&\quad = \left(b^+\sqrt{g(\rho-\rho_1)}G^{(0)}B_0^{-1}G_{11}^{(0)} + a^+\sqrt{g\rho_1}G^{(0)}B_0^{-1}G_{12}^{(0)}\right)\zeta\\
&\qquad - \left(b^-\sqrt{g(\rho-\rho_1)}G^{(0)}B_0^{-1}G_{11}^{(0)} + a^-\sqrt{g\rho_1}G^{(0)}B_0^{-1}G_{12}^{(0)}\right)\zeta_1\\
&\quad =: \mathcal{A}_1\zeta - \mathcal{B}_1\zeta_1, \qquad (5.23)
\end{aligned}$$

where symbols $\mathcal{A}_1$ and $\mathcal{B}_1$ are defined as

$$\begin{aligned}
\mathcal{A}_1 &:= b^+\sqrt{g(\rho-\rho_1)}G^{(0)}B_0^{-1}G_{11}^{(0)} + a^+\sqrt{g\rho_1}G^{(0)}B_0^{-1}G_{12}^{(0)},\\
\mathcal{B}_1 &:= b^-\sqrt{g(\rho-\rho_1)}G^{(0)}B_0^{-1}G_{11}^{(0)} + a^-\sqrt{g\rho_1}G^{(0)}B_0^{-1}G_{12}^{(0)}. \qquad (5.24)
\end{aligned}$$

Recalling the relation in Equation (4.26)

$$\eta = \frac{1}{\sqrt{g(\rho-\rho_1)}}(b^+\mu - b^-\mu_1), \qquad (5.25)$$

and applying Equations (5.22) and (5.23), we have

$$\begin{aligned}
R_1 &= -\frac{\rho-\rho_1}{2}\int_{\mathbb{R}} \eta\left(G^{(0)}B_0^{-1}(G_{11}^{(0)}\xi - G_{12}^{(0)}\xi_1)\right)^2 dx\\
&= -\frac{\rho-\rho_1}{2\sqrt{g(\rho-\rho_1)}}\int_{\mathbb{R}}(b^+\mu - b^-\mu_1)\left(\mathcal{A}_1\zeta - \mathcal{B}_1\zeta_1\right)^2. \qquad (5.26)
\end{aligned}$$

Similarly, for R_2, we calculate

$$G^{(0)}G_{12}^{(0)}B_0^{-1}\xi - \frac{1}{\rho_1}G^{(0)}B_0^{-1}(\rho G^{(0)} + \rho_1 G_{11}^{(0)})\xi_1 =: \mathcal{A}_2\zeta - \mathcal{B}_2\zeta_1, \qquad (5.27)$$

for symbols $\mathcal{A}_2$ and $\mathcal{B}_2$

$$\begin{aligned}
\mathcal{A}_2 &:= \frac{1}{\sqrt{g\rho_1}}a^+gG^{(0)}B_0^{-1}(\rho_1 G_{11}^{(0)} + \rho G^{(0)}) + b^+\sqrt{g(\rho-\rho_1)}G^{(0)}B_0^{-1}G_{12}^{(0)},\\
\mathcal{B}_2 &:= \frac{1}{\sqrt{g\rho_1}}a^-gG^{(0)}B_0^{-1}(\rho_1 G_{11}^{(0)} + \rho G^{(0)}) + b^-\sqrt{g(\rho-\rho_1)}G^{(0)}B_0^{-1}G_{12}^{(0)}.
\end{aligned} \qquad (5.28)$$

From Equations (4.26), (5.22) and (5.27), we have

$$\begin{aligned} R_2 &= -\frac{\rho_1}{2}\int_{\mathbb{R}} \eta_1 \left(G_{12}^{(0)} G^{(0)} B_0^{-1}\xi - \frac{1}{\rho_1}(G_{11}^{(0)}\xi_1 - \rho (G_{12}^{(0)})^2 B_0^{-1}\xi_1) \right)^2 dx \\ &= -\frac{\rho_1}{2\sqrt{g\rho_1}}\int_{\mathbb{R}} (a^-\mu_1 - a^+\mu)\left(\mathcal{A}_2\zeta - \mathcal{B}_2\zeta_1\right)^2 dx. \end{aligned} \tag{5.29}$$

Next, for R_3, we write

$$\begin{aligned} &DB_0^{-1}(G_{11}^{(0)}\xi - G_{12}^{(0)}\xi_1) \\ &\quad= \left(b^+\sqrt{g(\rho-\rho_1)}DB_0^{-1}G_{11}^{(0)} + a^+\sqrt{g\rho_1}DB_0^{-1}G_{12}^{(0)} \right)\zeta \\ &\qquad - \left(b^-\sqrt{g(\rho-\rho_1)}DB_0^{-1}G_{11}^{(0)} + a^-\sqrt{g\rho_1}DB_0^{-1}G_{12}^{(0)} \right)\zeta_1 \\ &\quad=: \mathcal{A}_3\zeta - \mathcal{B}_3\zeta_1, \end{aligned} \tag{5.30}$$

where symbols $\mathcal{A}_3$ and $\mathcal{B}_3$ are

$$\begin{aligned} \mathcal{A}_3 &= b^+\sqrt{g(\rho-\rho_1)}DB_0^{-1}G_{11}^{(0)} + a^+\sqrt{g\rho_1}DB_0^{-1}G_{12}^{(0)}, \\ \mathcal{B}_3 &= b^-\sqrt{g(\rho-\rho_1)}DB_0^{-1}G_{11}^{(0)} + a^-\sqrt{g\rho_1}DB_0^{-1}G_{12}^{(0)}. \end{aligned} \tag{5.31}$$

From Equations (4.26), (5.22) and (5.30), we also write

$$\begin{aligned} R_3 &= -\frac{\rho}{2}\int_{\mathbb{R}} \eta \left(DB_0^{-1}(G_{11}^{(0)}\xi - G_{12}^{(0)}\xi_1) \right)^2 dx \\ &= -\frac{\rho}{2\sqrt{g(\rho-\rho_1)}}\int_{\mathbb{R}} (b^+\mu - b^-\mu_1)\left(\mathcal{A}_3\zeta - \mathcal{B}_3\zeta_1\right)^2 dx. \end{aligned} \tag{5.32}$$

We have, for R_4, that

$$\begin{aligned} &DG^{(0)}B_0^{-1}\xi + \frac{\rho}{\rho_1}DG_{12}^{(0)}B_0^{-1}\xi_1 \\ &\quad= \left(b^+\sqrt{g(\rho-\rho_1)}DB_0^{-1}G^{(0)} - \frac{\rho}{\rho_1}a^+\sqrt{g\rho_1}DB_0^{-1}G_{12}^{(0)} \right)\zeta \\ &\qquad - \left(b^-\sqrt{g(\rho-\rho_1)}DB_0^{-1}G^{(0)} - \frac{\rho}{\rho_1}a^-\sqrt{g\rho_1}DB_0^{-1}G_{12}^{(0)} \right)\zeta_1 \\ &\quad=: \mathcal{A}_4\zeta - \mathcal{B}_4\zeta_1, \end{aligned} \tag{5.33}$$

where symbols $\mathcal{A}_4$ and $\mathcal{B}_4$ are defined as

$$\begin{aligned}\mathcal{A}_4 &:= b^+\sqrt{g(\rho-\rho_1)}DB_0^{-1}G^{(0)} - \frac{\rho}{\rho_1}a^+\sqrt{g\rho_1}DB_0^{-1}G_{12}^{(0)},\\ \mathcal{B}_4 &:= b^-\sqrt{g(\rho-\rho_1)}DB_0^{-1}G^{(0)} - \frac{\rho}{\rho_1}a^-\sqrt{g\rho_1}DB_0^{-1}G_{12}^{(0)}.\end{aligned} \tag{5.34}$$

Again applying Equations (4.26), (5.22) and (5.33), we calculate

$$\begin{aligned}R_4 &= \frac{\rho_1}{2}\int_{\mathbb{R}} \eta\left(DB_0^{-1}G^{(0)}\xi + \frac{\rho}{\rho_1}DG_{12}^{(0)}B_0^{-1}\xi_1\right)^2 dx\\ &= \frac{\rho_1}{2\sqrt{g(\rho-\rho_1)}}\int_{\mathbb{R}}(b^+\mu - b^-\mu_1)\left(\mathcal{A}_4\zeta - \mathcal{B}_4\zeta_1\right)^2 dx.\end{aligned} \tag{5.35}$$

Lastly, for R_5, we write

$$D\xi_1 =: \mathcal{A}_5\zeta - \mathcal{B}_5\zeta_1, \tag{5.36}$$

where, in particular, we have

$$\mathcal{A}_5 = -\sqrt{g\rho_1}a^+D, \qquad \mathcal{B}_5 = -\sqrt{g\rho_1}a^-D. \tag{5.37}$$

From Equations (4.26), (5.22) and (5.36), we deduce

$$\begin{aligned}R_5 &= -\frac{1}{2\rho_1}\int_{\mathbb{R}}\eta_1(D\xi_1)^2\,dx\\ &= \frac{1}{2\rho_1\sqrt{g\rho_1}}\int_{\mathbb{R}}(a^+\mu - a^-\mu_1)\left(\mathcal{A}_5\zeta - \mathcal{B}_5\zeta_1\right)^2 dx.\end{aligned} \tag{5.38}$$

Proposition 5.5. The cubic part of the Hamiltonian is

$$\begin{aligned}H^{(3)} = &-\frac{\rho-\rho_1}{2\sqrt{g(\rho-\rho_1)}}\int_{\mathbb{R}}(b^+\mu - b^-\mu_1)\left(\mathcal{A}_1\zeta - \mathcal{B}_1\zeta_1\right)^2 dx\\ &- \frac{\rho_1}{2\sqrt{g\rho_1}}\int_{\mathbb{R}}(a^-\mu_1 - a^+\mu)\left(\mathcal{A}_2\zeta - \mathcal{B}_2\zeta_1\right)^2 dx\\ &- \frac{\rho}{2\sqrt{g(\rho-\rho_1)}}\int_{\mathbb{R}}(b^+\mu - b^-\mu_1)\left(\mathcal{A}_3\zeta - \mathcal{B}_3\zeta_1\right)^2 dx\\ &+ \frac{\rho_1}{2\sqrt{g(\rho-\rho_1)}}\int_{\mathbb{R}}(b^+\mu - b^-\mu_1)\left(\mathcal{A}_4\zeta - \mathcal{B}_4\zeta_1\right)^2 dx\\ &+ \frac{1}{2\rho_1\sqrt{g\rho_1}}\int_{\mathbb{R}}(a^+\mu - a^-\mu_1)\left(\mathcal{A}_5\zeta - \mathcal{B}_5\zeta_1\right)^2 dx.\end{aligned} \tag{5.39}$$

Proof. This follows by adding together the contributions for each R_j from Equations (5.26), (5.29), (5.32), (5.35) and (5.38). □

6 THE HAMILTONIAN IN RESCALED COORDINATES

6.1 MULTIPLE SCALE ANALYSIS

In this section, we review multiple scale analysis, specifically lemmas on scale separation from [CGNS05], scaling transformation from [CGK05] and action on multiple scales from [CSS92]. These results measure the effect of Fourier multiplier operators, and by extension Dirichlet-Neumann operators, on long-wave scaling and they will be useful in representing the cubic terms of the Hamiltonian from Equation (5.39) in rescaled coordinates.

One may interpret the following scale separation lemma as the homogenisation of the fast oscillations due to short scale x at order $\mathcal{O}(\varepsilon^N)$. We say that the short and long scales, x and X, are asymptotically separated.

Lemma 6.1. *[CGNS05] (Scale Separation Lemma) Let $g(x)$ be a continuous function that is periodic on the fundamental domain $\mathbb{R}/\mathbb{Z}$. Then, for any function $f(X)$ of Schwartz class $\mathcal{S}(\mathbb{R})$ and for all N, we have*

$$\int_{\mathbb{R}} g(x)f(\varepsilon x)\,dx = \frac{1}{\varepsilon}\left(\int_0^1 g(x)\,dx\right)\int_{\mathbb{R}} f(X)\,dX + \mathcal{O}(\varepsilon^N). \tag{6.1}$$

Proof. This is proved in Lemma 3.2 of [CGNS05]. □

The spatial scaling transformation introduces the parameter ε into the Hamiltonian primarily through its effect on Fourier multiplier operators $m(D)$, which we recall is defined as

$$\begin{aligned} m(D_x)(f(x)) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ikx}m(k)(\mathcal{F}_x f)(k)\,dk \\ &= \frac{1}{2\pi}\iint\limits_{\mathbb{R}^2} e^{ik(x-x')}m(k)f(x')\,dx'\,dk. \end{aligned} \tag{6.2}$$

With the above definition, the next lemma describes the transformed Fourier multiplier after scaling.

Lemma 6.2. *[CGK05] Define $\widetilde{f}(X) = f(x)$ as the scaling transformation on $X = \varepsilon x$. Then the transformed Fourier multiplication operator is*

$$(m(D_x)f)(x) = \left(m(\varepsilon D_X)\widetilde{f}\right)(X). \tag{6.3}$$

Proof. Using the expression for the Fourier multiplier, we calculate

$$\begin{aligned}(m(D_x)f)(x) &= \frac{1}{2\pi}\iint_{\mathbb{R}^2} e^{ik(x-x')}m(k)f(x')\,dx'\,dk \\ &= \frac{1}{2\pi}\iint_{\mathbb{R}^2} \frac{1}{\varepsilon}e^{\frac{ik(X-X')}{\varepsilon}}m(k)f(X'/\varepsilon)\,dX'\,dk \\ &= \frac{1}{2\pi}\iint_{\mathbb{R}^2} e^{iK(X-X')}m(\varepsilon K)\widetilde{f}(X')\,dX'\,dK \\ &= \left(m(\varepsilon D_X)\widetilde{f}\right)(X), \end{aligned} \tag{6.4}$$

which yields the result. □

Next we state a lemma on the action of Fourier multiplier on functions of multiple scales. It is proven in Theorem 4.1 and extended in Appendix A2.1 of Craig, Sulem and Sulem [CSS92] for general pseudo-differential operators. The result assumes that the Fourier multipliers have the property that, for all $\gamma \in \mathbb{Z}^+$,

$$|\partial_k^j m(k)| \lesssim (1+|k|^2)^{\frac{\gamma-j}{2}} \tag{6.5}$$

for each $0 \leqslant j \leqslant \gamma$.

In the following formulation, $m(D_x)$ acts on the monochromatic oscillatory form $e^{ik_0x}\widetilde{f}(X)$ with a resulting Taylor expansion about wavenumber k_0. When truncated at order $\mathcal{O}(\varepsilon^N)$, it acts as a differential operator of order N.

Lemma 6.3. *[CSS92] Let $m(D)$ be a Fourier multiplier as defined in Equation (6.2) that satisfies Equation (6.5). Then, its action on multiple scales is*

$$(m(D_x)(e^{ik_0x}\widetilde{f}(X)) = e^{ik_0x}\left(m(k_0+\varepsilon D_X)\widetilde{f}\right)(X), \tag{6.6}$$

and has an asymptotic expansion

$$(m(D_x)(e^{ik_0x}\widetilde{f}(X)) = e^{ik_0x}\Big(\sum_{j=0}^{N}\frac{\varepsilon^j}{j!}m^{(j)}(k_0)D_X^j\widetilde{f}(X)\Big) + \mathcal{O}(\varepsilon^{N+1}). \tag{6.7}$$

Proof. We calculate

$$
\begin{aligned}
(m(D_x)(e^{ik_0x}\widetilde{f}(X)) &= \frac{1}{2\pi}\iint\limits_{\mathbb{R}^2} e^{ik(x-x')+ik_0x'}m(k)\widetilde{f}(\varepsilon x')\,dx'\,dk \\
&= \frac{e^{ik_0x}}{2\pi}\iint\limits_{\mathbb{R}^2} \frac{1}{\varepsilon}e^{\frac{ik'(X-X')}{\varepsilon}}m(k_0+k')\widetilde{f}(X')\,dX'\,dk' \\
&= \frac{e^{ik_0x}}{2\pi}\iint\limits_{\mathbb{R}^2} e^{iK''(X-X')}m(k_0+\varepsilon K')\widetilde{f}(X')\,dX'\,dK' \\
&= e^{ik_0x}\left(m(k_0+\varepsilon D_X)\widetilde{f}\right)(X).
\end{aligned}
\tag{6.8}
$$

Applying the approach in Theorem 4.1 of [CSS92], it then follows that

$$
\begin{aligned}
&\left(m(k_0+\varepsilon D_X)\widetilde{f}\right)(X) \\
&= \frac{1}{2\pi}\iint\limits_{\mathbb{R}^2} e^{iK(X-X')}m(k_0+\varepsilon K)\widetilde{f}(X')\,dX'\,dK \\
&= \frac{1}{2\pi}\iint\limits_{\mathbb{R}^2} e^{iK(X-X')}\left(\sum_{j=0}^{N}\frac{1}{j!}(m^{(j)})(k_0)(\varepsilon K)^j + R_{N+1}(\varepsilon K)\right)\widetilde{f}(X')\,dX'\,dK \\
&= \sum_{j=0}^{N}\frac{\varepsilon^j}{j!}m^{(j)}(k_0)D_X^j\widetilde{f}(X) + \mathcal{O}(\varepsilon^{N+1}),
\end{aligned}
\tag{6.9}
$$

since the Taylor remainder term is of order

$$
R_{N+1}(\varepsilon K) = \int_0^1 \frac{(1-t)^N}{N!}m^{(N+1)}(k_0+t\varepsilon K)(\varepsilon K)^{N+1}\,dt \sim \mathcal{O}(\varepsilon^{N+1}), \tag{6.10}
$$

which completes the proof. □

6.2 LONG-WAVE SCALING AND MODULATIONAL ANSATZ

We assume the scaling regime

$$
X = \varepsilon x, \qquad \varepsilon\widetilde{\mu}(X) = \mu(x), \qquad \widetilde{\zeta}(X) = \zeta(x) \tag{6.11}
$$

to describe the internal modes of the long waves. We prescribe asymptotics

$$
\frac{a}{h_1} \simeq \frac{h_1}{\lambda} \simeq \varepsilon \ll 1, \qquad \varepsilon_1 := \varepsilon^{1+\delta} \ll 1,
$$

where a is are the amplitude of the internal wave, λ is the wavelength of the surface wave and k_0 the wavenumber of the carrier wave. We will also choose $0 < \delta < \frac{1}{2}$.

We express the surface modes as quasi-monochromatic waves, which obey a modulational Ansatz given after transformation by

$$\begin{cases} \mu_1(x,t) &= \frac{\varepsilon_1}{\sqrt{2}}\omega_1^{1/2}(D_x)(v_1(X,t)e^{ik_0x} + \overline{v_1}(X,t)e^{-ik_0x}) \\ \zeta_1(x,t) &= \frac{\varepsilon_1}{\sqrt{2}i}\omega_1^{-1/2}(D_x)(v_1(X,t)e^{ik_0x} - \overline{v_1}(X,t)e^{-ik_0x}) \end{cases} \tag{6.12}$$

in new coordinates $(\mu_1, \zeta_1, v_1, \overline{v_1})$, where v_1 represents the envelope of the surface mode.

6.2.1 *Asymptotic Expansions of Symbols near $k = 0$*

Using Lemma 6.2, we find the asymptotic expansion of ω^2

$$\begin{aligned} \omega^2(\varepsilon D_X) &= \frac{g(1-\gamma)|\varepsilon D_X|\tanh(h_1|\varepsilon D_X|)}{1+\gamma\tanh(h_1\varepsilon D_X)} \\ &= g(1-\gamma)|\varepsilon D_X|\left(h_1|\varepsilon D_X| - \frac{1}{3}(h_1|\varepsilon D_X|)^3 + \mathcal{O}(\varepsilon^5)\right) \\ &\quad \cdot \left(1 - \gamma h_1|\varepsilon D_X| + \gamma^2 h_1^2|\varepsilon D_X|^2 - \gamma\big(\gamma^2 - \frac{1}{3}\big)h^3|\varepsilon D_X|^3 + \mathcal{O}(\varepsilon^4)\right) \\ &= \varepsilon^2 g(1-\gamma)h_1|D_X|^2 - \varepsilon^3 g\gamma(1-\gamma)h_1^2|D_X|^3 \\ &\quad + \varepsilon^4 g(1-\gamma)(\gamma^2 - \frac{1}{3})h_1^3|D_X|^4 + \mathcal{O}(\varepsilon^5) \\ &=: \varepsilon^2\frac{(\omega^2)^{(2)}}{2}D_X^2 + \varepsilon^3\frac{(\omega^2)^{(3)}}{6}D_X^2|D_X| + \varepsilon^4\frac{(\omega^2)^{(4)}}{24}D_X^4 \\ &\quad + \mathcal{O}(\varepsilon^5), \end{aligned} \tag{6.13}$$

where $\gamma = \frac{\rho_1}{\rho}$ is the density ratio. The first three coefficients in the asymptotic expansion of $\omega^2(\varepsilon D_X)$ are

$$\begin{cases} (\omega^2)^{(2)} &= 2g(1-\gamma)h_1 \\ (\omega^2)^{(3)} &= -6g\gamma(1-\gamma)h_1^2 \\ (\omega^2)^{(4)} &= 24g(1-\gamma)\big(\gamma^2 - \frac{1}{3}\big)h_1^3. \end{cases} \tag{6.14}$$

Next, from Equation (4.25), we have

$$a^{+}(D) = \frac{-\sqrt{\rho-\rho_1}G_{12}^{(0)}}{\sqrt{\left(\rho G_{11}^{(0)} + (2\rho_1-\rho)G^{(0)}\right)\left(G_{11}^{(0)} + G^{(0)}\right)}} \tag{6.15}$$

and the Taylor expansion of the symbol is

$$a^{+}(k) = \sqrt{1-\gamma}\frac{\operatorname{csch}(h_1k)}{\coth(h_1k)} - \gamma\sqrt{1-\gamma}\frac{\operatorname{csch}(h_1k)}{k\coth^2(h_1k)}|k| + \mathcal{O}(|k|^2). \tag{6.16}$$

By Lemma 6.2, when $a^{+}(D)$ is applied to the long-wave function $\mu(x) = \varepsilon\widetilde{\mu}(X)$, where $X = \varepsilon x$,

$$\begin{aligned} a^{+}(D_x)\mu &= \varepsilon a^{+}(\varepsilon D_X)\widetilde{\mu}(X) \\ &= \varepsilon\sqrt{1-\gamma}\widetilde{\mu} - \varepsilon^2\gamma\sqrt{1-\gamma}h_1|D_X|\widetilde{\mu} + \mathcal{O}(\varepsilon^3) \\ &=: \varepsilon\big((a^{+})^{(0)} + \varepsilon(a^{+})^{(1)}|D_X|\big)\widetilde{\mu} + \mathcal{O}(\varepsilon^3). \end{aligned} \tag{6.17}$$

Similarly, we have from Lemma 4.4,

$$b^{+}(D) = \frac{\rho_1}{\rho}\left(\frac{\mathcal{Q}_a(D) + \mathcal{Q}_c(D)}{\mathcal{Q}_b(D)}\right)a^{+}(D) = \frac{\rho_1(G_{11}^{(0)} + G^{(0)})}{\sqrt{\rho_1(\rho-\rho_1)}G_{12}^{(0)}}a^{+}(D), \tag{6.18}$$

and thus $b^{+}(D)$ acting on $\widetilde{\mu}(X)$ is

$$\begin{aligned} b^{+}(D_x)\mu &= \varepsilon b^{+}(\varepsilon D_X)\widetilde{\mu}(X) \\ &= \varepsilon\sqrt{\frac{\gamma}{1-\gamma}}(1 + h_1\varepsilon|D_X|)\sqrt{1-\gamma}(1-\gamma\varepsilon h_1|D_X|) + \mathcal{O}(\varepsilon^3) \\ &= \varepsilon\sqrt{\gamma}\widetilde{\mu} + \varepsilon^2\sqrt{\gamma}(1-\gamma)h_1|D_X|\widetilde{\mu} + \mathcal{O}(\varepsilon^3) \\ &=: \varepsilon\big((b^{+})^{(0)} + \varepsilon(b^{+})^{(1)}|D_X|\big)\widetilde{\mu} + \mathcal{O}(\varepsilon^3). \end{aligned} \tag{6.19}$$

The coefficients defined in Equations (6.17) and (6.19) are

$$\begin{cases} (a^{+})^{(0)} = \sqrt{1-\gamma}, & (a^{+})^{(1)} = -\gamma\sqrt{1-\gamma}h_1 \\ (b^{+})^{(0)} = \sqrt{\gamma}, & (b^{+})^{(1)} = \sqrt{\gamma}(1-\gamma)h_1. \end{cases} \tag{6.20}$$

6.2.2 *Quadratic Terms of Hamiltonian in Rescaled Coordinates*

The next step is to substitute these scalings and the modulational Ansatz and perform the expansions of the Hamiltonian. This involves first replacing multiplier $\omega(D)$ acting on long-scale functions $\widetilde{f}(X) = f(\varepsilon x)$ with

$\omega(\varepsilon D_X)$ and $\omega_1(D)$ acting on multiple scale functions of x and X with $\omega_1(k_0+\varepsilon D_X)$. Second, one finds the Taylor expansion of these Fourier multipliers using Lemma 6.3 and then, third, truncates at order $\mathcal{O}(\varepsilon^4)$ to obtain the expansion of the quadratic part of the Hamiltonian.

Now we write the leading, quadratic terms $H^{(2)}$ in the Hamiltonian in rescaled variables $(\tilde{\zeta},\tilde{\mu},v_1,\overline{v_1})$ from our Benjamin-Ono and modulational regimes.

Lemma 6.4. *The quadratic part of the Hamiltonian in rescaled coordinates is*

$$\begin{aligned} H^{(2)} = \int_{\mathbb{R}} \Big[&-\varepsilon\frac{(\omega^2)^{(2)}}{4}(D_X\tilde{\zeta})^2 + \varepsilon^2\frac{(\omega^2)^{(3)}}{12}\tilde{\zeta}(D_X^2|D_X|\tilde{\zeta}) \\ &+\varepsilon^3\frac{(\omega^2)^{(4)}}{48}\tilde{\zeta}(D_X^4\tilde{\zeta}) + \varepsilon\tilde{\mu}^2 + \frac{\varepsilon_1^2}{\varepsilon}\omega_1(k_0)|v_1|^2 \\ &+\varepsilon_1^2\omega_1'(k_0)\overline{v_1}(D_Xv_1) + \varepsilon\varepsilon_1^2\frac{\omega_1''(k_0)}{2}\overline{v_1}(D_X^2v_1)\Big]\,dX + \mathcal{O}(\varepsilon^4). \end{aligned} \tag{6.21}$$

Proof. Starting from the quadratic terms of the Hamiltonian $H^{(2)}$ given in Equation (4.47), and applying Lemma 6.3 and Equation (6.14), we calculate

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}}\left[\zeta\omega^2(D_x)\zeta+\mu^2\right]dx \\ &=\frac{1}{2}\int_{\mathbb{R}}\left[\tilde{\zeta}\omega^2(\varepsilon D_X)\tilde{\zeta}+(\varepsilon\tilde{\mu})^2\right]\frac{dX}{\varepsilon} \\ &=\frac{1}{2}\int_{\mathbb{R}}\Big[\varepsilon^{-1}\tilde{\zeta}\Big(\frac{\varepsilon^2}{2}(\omega^2)^{(2)}D_X^2+\frac{\varepsilon^3}{6}(\omega^2)^{(3)}D_X^2|D_X| \\ &\qquad+\frac{\varepsilon^4}{24}(\omega^2)^{(4)}D_X^4\Big)\tilde{\zeta}+\varepsilon\tilde{\mu}^2\Big]\,dX+\mathcal{O}(\varepsilon^4) \\ &=\int_{\mathbb{R}}\Big[\varepsilon\frac{(\omega^2)^{(2)}}{4}\tilde{\zeta}(D_X^2\tilde{\zeta})+\varepsilon^2\frac{(\omega^2)^{(3)}}{12}\tilde{\zeta}(D_X^2|D_X|\tilde{\zeta}) \\ &\qquad+\varepsilon^3\frac{(\omega^2)^{(4)}}{48}\tilde{\zeta}(D_X^4\tilde{\zeta})+\varepsilon\tilde{\mu}^2\Big]\,dX+\mathcal{O}(\varepsilon^4) \\ &=\int_{\mathbb{R}}\Big[-\varepsilon\frac{(\omega^2)^{(2)}}{4}(D_X\tilde{\zeta})^2+\varepsilon^2\frac{(\omega^2)^{(3)}}{12}\tilde{\zeta}(D_X^2|D_X|\tilde{\zeta}) \\ &\qquad+\varepsilon^3\frac{(\omega^2)^{(4)}}{48}\tilde{\zeta}(D_X^4\tilde{\zeta})+\varepsilon\tilde{\mu}^2\Big]\,dX+\mathcal{O}(\varepsilon^4). \end{aligned} \tag{6.22}$$

Next, applying Lemmas 6.1 and 6.3, as well as our modulational Ansatz from Equation (6.12), we calculate

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}} \left[\zeta_1 \omega_1^2(D_x)\zeta_1 + \mu_1^2\right] dx \\
&\quad = \frac{1}{2}\int_{\mathbb{R}} \Bigg[-\frac{\varepsilon_1^2}{2}\left(v_1 e^{ik_0x} - \overline{v_1}e^{-ik_0x}\right)\left(\omega_1(D_x)(v_1 e^{ik_0x} - \overline{v_1}e^{-ik_0x})\right) \\
&\qquad + \frac{\varepsilon_1^2}{2}\left(v_1 e^{ik_0x} + \overline{v_1}e^{-ik_0x}\right)\left(\omega_1(D_x)(v_1 e^{ik_0x} + \overline{v_1}e^{-ik_0x})\right)\Bigg] \frac{dX}{\varepsilon} \\
&\quad = \frac{1}{2}\int_{\mathbb{R}} \Bigg[-\frac{\varepsilon_1^2}{2\varepsilon}\big(- v_1\omega_1(-k_0+\varepsilon D_X)\overline{v_1} - \overline{v_1}\omega_1(k_0+\varepsilon D_X)v_1\big) \\
&\qquad + \frac{\varepsilon_1^2}{\varepsilon}\big(v_1\omega_1(-k_0+\varepsilon D_X)\overline{v_1} + \overline{v_1}\omega_1(k_0+\varepsilon D_X)v_1\big)\Bigg] dX + \mathcal{O}(\varepsilon_1^2\varepsilon^N) \\
&\quad = \frac{1}{2}\int_{\mathbb{R}} \Bigg[\frac{2\varepsilon_1^2}{\varepsilon} v_1\big(\omega_1(-k_0) + \varepsilon\omega_1'(-k_0)D_X + \frac{\varepsilon^2}{2}\omega_1''(-k_0)D_X^2\big)\overline{v_1} \\
&\qquad + \frac{\varepsilon_1^2}{\varepsilon}\overline{v_1}\big(\omega_1(k_0) + \varepsilon\omega_1'(k_0)D_X + \frac{\varepsilon^2}{2}\omega_1''(k_0)D_X^2\big)v_1\Bigg] dX + \mathcal{O}(\varepsilon^2\varepsilon_1^2) \\
&\quad = \int_{\mathbb{R}} \left[\frac{\varepsilon_1^2}{\varepsilon}\omega_1^2(k_0)|v_1|^2 + \varepsilon_1^2\omega_1'(k_0)\overline{v_1}(D_X v_1) + \frac{\varepsilon\varepsilon_1^2\omega_1''(k_0)}{2}\overline{v_1}(D_X^2 v_1)\right] dX \\
&\qquad + \mathcal{O}(\varepsilon^2\varepsilon_1^2).
\end{aligned} \tag{6.23}
$$

Combining Equations (6.22) and (6.23), we get the result. □

6.2.3 *The Momentum in Rescaled Coordinates*

We calculate the momentum I in rescaled coordinates $(\widetilde{\zeta}, \widetilde{\mu}, v_1, \overline{v_1})$.

Lemma 6.5. *The momentum I in rescaled coordinates is*

$$
\begin{aligned}
I = &-\int_{\mathbb{R}} i\varepsilon\widetilde{\zeta}(D_X\widetilde{\mu}) - \frac{\varepsilon_1^2}{\varepsilon}k_0|v_1|^2 - \frac{\varepsilon_1^2}{2}\left[(D_X v_1)\overline{v_1} + v_1(\overline{D_X v_1})\right] dX \\
&+ \mathcal{O}(\varepsilon_1^2\varepsilon^N).
\end{aligned} \tag{6.24}
$$

Proof. Applying Lemmas 6.1 and our modulational Ansatz from Equation (6.12), we calculate

$$\begin{aligned} I &= -\int_{\mathbb{R}} (\zeta \partial_x \mu + \zeta_1 \partial_x \mu_1)\, dx \\ &= -\int_{\mathbb{R}} \left[\varepsilon \widetilde{\zeta}(\partial_X \widetilde{\mu}) - \frac{\varepsilon_1^2}{\varepsilon} k_0 |v_1|^2 + \frac{\varepsilon_1^2}{2i} \left[v_1 (\partial_X \overline{v_1}) - \overline{v_1}(\partial_X v_1) \right] \right] dX + \mathcal{O}(\varepsilon_1^2 \varepsilon^N) \\ &= -\int_{\mathbb{R}} i\varepsilon \widetilde{\zeta}(D_X \widetilde{\mu}) - \frac{\varepsilon_1^2}{\varepsilon} k_0 |v_1|^2 - \frac{\varepsilon_1^2}{2} \left[(D_X v_1)\overline{v_1} + v_1 (\overline{D_X v_1}) \right] dX \\ &\quad + \mathcal{O}(\varepsilon_1^2 \varepsilon^N), \end{aligned} \tag{6.25}$$

completing the proof. □

6.3 CUBIC TERMS OF THE HAMILTONIAN IN RESCALED VARIABLES

In view of Lemmas 6.1, 6.2 and 6.3 from the section on multiple scales, we focus on calculating the relevant terms in the cubic terms of the Hamiltonian $H^{(3)}$ in terms of rescaled variables $(\widetilde{\mu}, \widetilde{\zeta}, v_1, \overline{v_1})$, which are later used in the derivation of the Benjamin-Ono and Schrödinger coupled system. In this section, we prove the following proposition.

Proposition 6.6. The cubic part of the Hamiltonian $H^{(3)}$ in rescaled variables is

$$\begin{aligned} H^{(3)} = \int_{\mathbb{R}} \Big[& \varepsilon^2 \kappa \widetilde{\mu} (D_X \widetilde{\zeta})^2 + \varepsilon_1^2 (\kappa_1 \widetilde{\mu} + \kappa_2 \partial_X \widetilde{\zeta}) |v_1|^2 + \varepsilon^3 \kappa_3 \widetilde{\mu} (|D_X| D_X \widetilde{\zeta})(D_X \widetilde{\zeta}) \\ & + \varepsilon \varepsilon_1^2 (\kappa_4 \widetilde{\mu} + \kappa_5 (\partial_X \widetilde{\zeta})) \left[v_1 \overline{D_X v_1} + \overline{v_1}(D_X v_1) \right] \\ & + \varepsilon \varepsilon_1^2 \big(\kappa_6 (|D_X| \widetilde{\mu}) + \kappa_7 (|D_X| \partial_X \widetilde{\zeta}) \big) |v_1|^2 + \varepsilon^3 \kappa_8 (|D_X| \widetilde{\mu})(D_X \widetilde{\zeta})^2 \Big] \, dX \\ & + \mathcal{O}(\varepsilon^4), \end{aligned} \tag{6.26}$$

in rescaled variables $(\widetilde{\mu}, \widetilde{\zeta}, v_1, \overline{v_1})$, where κ and κ_j, for $j \in \{1, 2, 3, 4, 5, 6, 7, 8\}$, only depend on physical parameters g, h_1, ρ and ρ_1. Their precise expressions will be given in terms of heretofore undefined symbols in the proof at the end of Section 6.3.

6.3.1 *Relevant Cubic Terms of the Hamiltonian in Variables* $(\mu, \mu_1, \zeta, \zeta_1)$

The integrands of the relevant terms in Equation (5.39) are of the form

$$\begin{cases} (m_1(D_x)\mu)(m_2(D_x)\zeta)^2 \\ (m_1(D_x)\mu)(m_2(D_x)\zeta_1)^2 \\ (m_1(D_x)\mu_1)(m_2(D_x)\zeta)(m_3(D_x)\zeta_1) \end{cases} \tag{6.27}$$

as the other terms can be omitted by the following application of Lemma 6.1. More specifically, in the scaling regime given by Equations (6.11) and (6.12), the cubic terms of the Hamiltonian $H^{(3)}$ with integrands of the form

$$\begin{cases} (m_1(D_x)\mu_1)(m_2(D_x)\zeta)^2 \\ (m_1(D_x)\mu_1)(m_2(D_x)\zeta_1)^2 \\ (m_1(D_x)\mu)(m_2(D_x)\zeta)(m_3(D_x)\zeta_1) \end{cases} \tag{6.28}$$

are negligible, that is the integrals are of order $\lesssim \varepsilon^N$ for each N.

Applying Lemmas 6.1, 6.2 and 6.3, we calculate for all N that

$$\begin{aligned} &\int_{\mathbb{R}} (m_1(D_x)\mu_1)(m_2(D_x)\zeta)^2\, dx \\ &= \int_{\mathbb{R}} \left(\frac{\varepsilon_1}{\sqrt{2}} (m_1\omega_1^{1/2})(D_x)\big(v_1(X,t)e^{ik_0x} + \overline{v_1}(X,t)e^{-ik_0x}\big) \right) (m_2(D_x)\zeta)^2\, dx \\ &= \frac{\varepsilon_1}{\sqrt{2}} \int_{\mathbb{R}} e^{ik_0x} \Big((m_1\omega_1^{1/2})(k_0 + \varepsilon D_X) v_1(X,t) \Big) (m_2(\varepsilon D_X)\widetilde{\zeta})^2\, dx + \text{c.c.} \\ &= \frac{\varepsilon_1}{\sqrt{2}\varepsilon} \left(\frac{k_0}{2\pi} \int_0^{\frac{2\pi}{k_0}} e^{ik_0x}\, dx \right) \int_{\mathbb{R}} \Big((m_1\omega_1^{1/2})(k_0 + \varepsilon D_X) v_1 \Big) (m_2(\varepsilon D_X)\widetilde{\zeta})^2\, dX \\ &\quad + \text{c.c.} + \mathcal{O}(\varepsilon_1\varepsilon^N) \lesssim \mathcal{O}(\varepsilon_1\varepsilon^N). \end{aligned} \tag{6.29}$$

Similarly, we find the asymptotics of integrals of the second type

$$\begin{aligned}
&\int_{\mathbb{R}} (m_1(D_x)\mu_1)(m_2(D_x)\zeta_1)^2\,dx \\
&= \int_{\mathbb{R}} \left(\frac{\varepsilon_1}{\sqrt{2}} (m_1\omega_1^{1/2})(D_x)\big(v_1(X,t)e^{ik_0x} + \overline{v_1}(X,t)e^{-ik_0x}\big) \right) \\
&\cdot \left(\frac{\varepsilon_1}{\sqrt{2}i} (m_2\omega_1^{-1/2})(D_x)(v_1(X,t)e^{ik_0x} - \overline{v_1}(X,t)e^{-ik_0x}) \right)^2 dx \\
&= \frac{-\varepsilon_1^3}{2\sqrt{2}} \int_{\mathbb{R}} \Big(e^{ik_0x}\big((m_1\omega_1^{1/2})(k_0+\varepsilon D_X)v_1\big) + e^{-ik_0x}\big((m_1\omega_1^{1/2})(-k_0+\varepsilon D_X)\overline{v_1}\big)\Big) \\
&\cdot \Big(e^{ik_0x}\big((m_2\omega_1^{-1/2})(k_0+\varepsilon D_X)v_1\big) - e^{-ik_0x}\big((m_2\omega_1^{-1/2})(-k_0+\varepsilon D_X)\overline{v_1}\big)\Big)^2 dx \\
&\lesssim \mathcal{O}(\varepsilon_1\varepsilon^N), \qquad (6.30)
\end{aligned}$$

since after simplification the only exponential terms to appear in the integrand are $e^{\pm ik_0x}$ and $e^{\pm 3ik_0x}$. Lastly, we calculate

$$\begin{aligned}
&\int_{\mathbb{R}} (m_1(D_x)\mu)(m_2(D_x)\zeta)(m_3(D_x)\zeta_1)\,dx \\
&\quad= \int_{\mathbb{R}} (m_1(D_x)\mu)(m_2(D_x)\zeta) \\
&\quad\cdot \left(\frac{\varepsilon_1}{\sqrt{2}i} (m_3\omega_1)^{-1/2}(D_x)\big(v_1(X,t)e^{ik_0x} - \overline{v_1}(X,t)e^{-ik_0x}\big) \right) dx \\
&\quad= \frac{\varepsilon\varepsilon_1}{\sqrt{2}i} \int_{\mathbb{R}} e^{ik_0x}(m_1(\varepsilon D_X)\widetilde{\mu})(m_2(\varepsilon D_X)\widetilde{\zeta})\big((m_3\omega_1)^{-1/2}(k_0+\varepsilon D_X)v_1(X,t)\big) \\
&\quad+ \text{c.c.} \lesssim \mathcal{O}(\varepsilon_1\varepsilon^N). \qquad (6.31)
\end{aligned}$$

We have reduced the cubic part of the Hamiltonian to the relevant terms, where we have eliminated those that are negligible by Lemma 6.1. This is done methodically for each R_j, $j \in \{1,2,3,4,5\}$ as defined in Equation (5.22).

Corollary 6.7. *In the Benjamin-Ono scaling and modulational regime, the cubic part of the Hamiltonian $H^{(3)}$ given in Equation (5.5) is*

$$
\begin{aligned}
&H^{(3)} \\
&= \frac{-(\rho-\rho_1)}{2\sqrt{g(\rho-\rho_1)}} \int_{\mathbb{R}} \left[(b^+\mu)(\mathcal{A}_1\zeta)^2 + (b^+\mu)(\mathcal{B}_1\zeta_1)^2 + 2(b^-\mu_1)(\mathcal{A}_1\zeta)(\mathcal{B}_1\zeta_1)\right] dx \\
&+ \frac{\rho_1}{2\sqrt{g\rho_1}} \int_{\mathbb{R}} \left[(a^+\mu)(\mathcal{A}_2\zeta)^2 + (a^+\mu)(\mathcal{B}_2\zeta_1)^2 + 2(a^-\mu_1)(\mathcal{A}_2\zeta)(\mathcal{B}_2\zeta_1)\right] dx \\
&- \frac{\rho}{2\sqrt{g(\rho-\rho_1)}} \int_{\mathbb{R}} \left[(b^+\mu)(\mathcal{A}_3\zeta)^2 + (b^+\mu)(\mathcal{B}_3\zeta_1)^2 + 2(b^-\mu_1)(\mathcal{A}_3\zeta)(\mathcal{B}_3\zeta_1)\right] dx \\
&+ \frac{\rho_1}{2\sqrt{g(\rho-\rho_1)}} \int_{\mathbb{R}} \left[[(b^+\mu)(\mathcal{A}_4\zeta)^2 + (b^+\mu)(\mathcal{B}_4\zeta_1)^2 + 2(b^-\mu_1)(\mathcal{A}_4\zeta)(\mathcal{B}_4\zeta_1)\right] dx \\
&+ \frac{1}{2\rho_1\sqrt{g\rho_1}} \int_{\mathbb{R}} \left[(a^+\mu)(\mathcal{A}_5\zeta)^2 + (a^+\mu)(\mathcal{B}_5\zeta_1)^2 + 2(a^-\mu_1)(\mathcal{A}_5\zeta)(\mathcal{B}_5\zeta_1)\right] dx \\
&+ \text{h.o.t.} \\
&=: R_1 + R_2 + R_3 + R_4 + R_5 + \text{h.o.t.}
\end{aligned} \tag{6.32}
$$

in normal variables, where we now relabel each R_j term from Equation (5.22).

Proof. This follows directly from Proposition 5.5 and from our calculation of the negligible terms. □

6.3.2 *Action of Fourier Multipliers on Multiple Scale Functions*

By Corollary 6.7, the first, second and third terms in the integrand of each R_j in Equation (6.32) are of the form

$$
\begin{aligned}
&(\mathcal{P}(D)\mu)(\mathcal{Q}(D)\zeta)^2, \\
&(\mathcal{P}(D)\mu)(\mathcal{R}(D)\zeta_1)^2, \\
&(\mathcal{P}(D)\mu_1)(\mathcal{Q}(D)\zeta)(\mathcal{R}(D)\zeta_1),
\end{aligned} \tag{6.33}
$$

respectively, where $\mathcal{P}(D)$, $\mathcal{Q}(D)$ and $\mathcal{R}(D)$ are all Fourier multipliers. In the first two propositions, $\mathcal{P}(D)$ will be either $a^+(D)$ or $b^+(D)$, whose expansions at $k = 0$ are given in Equations (6.17) and (6.19). The following three propositions will allow us to expand, in powers of ε and ε_1, terms of each form that appears in Equation (6.33).

Proposition 6.8. Let $\mathcal{P}(D_x)$ and

$$\mathcal{Q}(D_x) = \mathcal{Q}^{(1)}(D_x)D_x + \mathcal{Q}^{(2)}(D_x)|D_x|D_x, \tag{6.34}$$

where

$$\mathcal{Q}^{(j)}(\varepsilon D_X) =: Q^{(j,0)} + \varepsilon Q^{(j,1)}|D_X| + \mathcal{O}(\varepsilon^2) \tag{6.35}$$

for $j \in \{0,1\}$, be Fourier multipliers. Then,

$$\begin{aligned}
&(\mathcal{P}(D_x)\mu)(\mathcal{Q}(D_x)\zeta)^2 \\
&\quad = \varepsilon^3 \mathcal{P}^{(0)}(\mathcal{Q}^{(1,0)})^2 \widetilde{\mu}(D_X\widetilde{\zeta})^2 \\
&\qquad + \varepsilon^4 \mathcal{P}^{(1)}(\mathcal{Q}^{(1,0)})^2(|D_X|\widetilde{\mu})\big(D_X\widetilde{\zeta}\big)^2 \\
&\qquad + 2\varepsilon^4 \mathcal{P}^{(0)}\mathcal{Q}^{(1,0)}(\mathcal{Q}^{(1,1)})\widetilde{\mu}(D_X\widetilde{\zeta})\big(|D_X|D_X\widetilde{\zeta}\big) \\
&\qquad + 2\varepsilon^4 \mathcal{P}^{(0)}\mathcal{Q}^{(1,0)}\mathcal{Q}^{(2,0)}\widetilde{\mu}(D_X\widetilde{\zeta})(|D_X|D_X\widetilde{\zeta}) + \mathcal{O}(\varepsilon^5).
\end{aligned} \tag{6.36}$$

Proof. By Lemmas 6.2 and 6.3, we expand

$$\begin{aligned}
&(\mathcal{P}(D_x)\mu)(\mathcal{Q}(D_x)\zeta)^2 \\
&\quad = \big(\varepsilon\mathcal{P}(\varepsilon D_X)\widetilde{\mu}\big)\big(\varepsilon\mathcal{Q}^{(1)}(\varepsilon D_X)D_X\widetilde{\zeta} + \varepsilon^2\mathcal{Q}^{(2)}(\varepsilon D_X)|D_X|D_X\widetilde{\zeta}\big)^2 \\
&\quad = \big(\varepsilon\mathcal{P}^{(0)}\widetilde{\mu} + \varepsilon^2\mathcal{P}^{(1)}|D_X|\widetilde{\mu}\big) \\
&\qquad \cdot \big(\varepsilon^2(\mathcal{Q}^{(1,0)})^2(D_X\widetilde{\zeta})^2 + 2\varepsilon^3\mathcal{Q}^{(1,0)}(D_X\widetilde{\zeta})\big(\mathcal{Q}^{(1,1)}|D_X|D_X\widetilde{\zeta}\big) \\
&\qquad + 2\varepsilon^3\mathcal{Q}^{(1,0)}\mathcal{Q}^{(2,0)}(D_X\widetilde{\zeta})(|D_X|D_X\widetilde{\zeta})\big) \\
&\qquad + \mathcal{O}(\varepsilon^5),
\end{aligned} \tag{6.37}$$

which after expansion yields the result. □

Proposition 6.9. Let $\mathcal{P}(D_x)$ and $\mathcal{R}(D_x)$ be Fourier multipliers with $\mathcal{R}$ either even or odd. Then,

$$\begin{aligned}
&(\mathcal{P}(D_x)\mu)(\mathcal{R}(D_x)\zeta_1)^2 \\
&\quad = (-1)^j\varepsilon\varepsilon_1^2\mathcal{P}^{(0)}(\mathcal{R}^2\omega_1^{-1})(k_0)\widetilde{\mu}|v_1|^2 \\
&\qquad + (-1)^j\varepsilon^2\varepsilon_1^2\mathcal{P}^{(1)}(\mathcal{R}^2\omega_1^{-1})(k_0)\big(|D_X|\widetilde{\mu}\big)|v_1|^2 \\
&\qquad + \frac{\varepsilon^2\varepsilon_1^2}{2}\mathcal{P}^{(0)}(\mathcal{R}^2\omega_1^{-1})'(k_0)\big[v_1(\overline{D_X v_1}) + (D_X v_1)\overline{v_1}\big] + \mathcal{O}(\varepsilon^3\varepsilon_1^2)
\end{aligned} \tag{6.38}$$

with $j = 0$ if $\mathcal{R}$ is even and $j = 1$ if $\mathcal{R}$ is odd.

Proof. By Lemmas 6.1, 6.2 and 6.3, we expand

$$
\begin{aligned}
&(\mathcal{P}(D_x)\mu)(\mathcal{R}(D_x)\zeta_1)^2\\
&\quad= \big(\varepsilon\mathcal{P}(\varepsilon D_X)\widetilde{\mu}\big)\Big(\frac{\varepsilon_1}{\sqrt{2}i}(\mathcal{R}\omega_1^{-1/2})(\varepsilon D_X)(v_1e^{ik_0x}-\overline{v_1}e^{-ik_0x})\Big)^2\\
&\quad= \frac{-\varepsilon\varepsilon_1^2}{2}\big(\mathcal{P}(\varepsilon D_X)\widetilde{\mu}\big)\big(e^{ik_0x}(\mathcal{R}\omega_1^{-1/2})(k_0+\varepsilon D_X)v_1\\
&\qquad-e^{-ik_0x}(\mathcal{R}\omega_1^{-1/2})(-k_0+\varepsilon D_X)\overline{v_1}\big)^2\\
&\quad= \varepsilon\varepsilon_1^2\big(\mathcal{P}(\varepsilon D_X)\widetilde{\mu}\big)\big((\mathcal{R}\omega_1^{-1/2})(k_0+\varepsilon D_X)v_1\big)\big((\mathcal{R}\omega_1^{-1/2})(-k_0+\varepsilon D_X)\overline{v_1}\big)\\
&\qquad+\mathcal{O}(\varepsilon^N)\\
&\quad= \varepsilon\varepsilon_1^2\big(\mathcal{P}^{(0)}\widetilde{\mu}+\varepsilon\mathcal{P}^{(1)}|D_X|\widetilde{\mu}\big)\\
&\qquad\cdot\big((\mathcal{R}\omega_1^{-1/2})(k_0)v_1+\varepsilon(\mathcal{R}\omega_1^{-1/2})'(k_0)D_Xv_1\big)\\
&\qquad\cdot\big((\mathcal{R}\omega_1^{-1/2})(-k_0)\overline{v_1}+\varepsilon(\mathcal{R}\omega_1^{-1/2})'(-k_0)D_X\overline{v_1}\big)+\mathcal{O}(\varepsilon^3\varepsilon_1^2),
\end{aligned}
\tag{6.39}
$$

which after expansion completes the proof. □

Proposition 6.10. Let $\mathcal{P}(D_x)$, $\mathcal{R}(D_x)$ and

$$\mathcal{Q}(D_x)=\mathcal{Q}^{(1)}(D_x)D_x+\mathcal{Q}^{(2)}(D_x)|D_x|D_x, \tag{6.40}$$

where

$$\mathcal{Q}^{(j)}(\varepsilon D_X)=: Q^{(j,0)}+\varepsilon Q^{(j,1)}|D_X|+\mathcal{O}(\varepsilon^2) \tag{6.41}$$

for $j\in\{0,1\}$, be Fourier multipliers with $\mathcal{P}$ even and $\mathcal{R}$ odd. Then,

$$
\begin{aligned}
&(\mathcal{P}(D_x)\mu_1)(\mathcal{Q}(D_x)\zeta)(\mathcal{R}(D_x)\zeta_1)\\
&\quad=-\varepsilon\varepsilon_1^2\mathcal{Q}^{(1,0)}(\mathcal{P}\mathcal{R})(k_0)(\partial_X\widetilde{\zeta})|v_1|^2-\varepsilon^2\varepsilon_1^2(\mathcal{Q}^{(1,1)})(\mathcal{P}\mathcal{R})(k_0)\big(|D_X|\partial_X\widetilde{\zeta}\big)|v_1|^2\\
&\qquad-\varepsilon^2\varepsilon_1^2\mathcal{Q}^{(2,0)}(\mathcal{P}\mathcal{R})(k_0)(|D_X|\partial_X\widetilde{\zeta})|v_1|^2\\
&\qquad-\frac{\varepsilon^2\varepsilon_1^2}{2}\mathcal{Q}^{(1,0)}(\mathcal{P}\mathcal{R})'(k_0)(\partial_X\widetilde{\zeta})\big[v_1(D_X\overline{v_1})+(D_Xv_1)\overline{v_1}\big]\\
&\qquad+\mathcal{O}(\varepsilon^3\varepsilon_1^2).
\end{aligned}
\tag{6.42}
$$

Proof. By Lemmas 6.1, 6.2 and 6.3, we expand

$$
\begin{aligned}
&(\mathcal{P}(D_x)\mu_1)(\mathcal{Q}(D_x)\zeta)(\mathcal{R}(D_x)\zeta_1)\\
&\quad= \big(\frac{\varepsilon_1}{\sqrt{2}}(\mathcal{P}\omega_1^{1/2})(\varepsilon D_X)(v_1 e^{ik_0x} + \overline{v_1}e^{-ik_0x})\big)\big(\mathcal{Q}(\varepsilon D_X)\tilde{\zeta}\big)\\
&\qquad\cdot\big(\frac{\varepsilon_1}{\sqrt{2}i}(\mathcal{R}\omega_1^{-1/2})(\varepsilon D_X)(v_1 e^{ik_0x} - \overline{v_1}e^{-ik_0x})\big)\\
&\quad= i\frac{\varepsilon_1^2}{2}\big((\mathcal{P}\omega_1^{1/2})(k_0+\varepsilon D_X)(v_1)\big)\big(\mathcal{Q}(\varepsilon D_X)\tilde{\zeta}\big)\big((\mathcal{R}\omega_1^{-1/2})(-k_0+\varepsilon D_X)(\overline{v_1})\big)\\
&\qquad- i\frac{\varepsilon_1^2}{2}\big((\mathcal{P}\omega_1^{1/2})(-k_0+\varepsilon D_X)(\overline{v_1})\big)\big(\mathcal{Q}(\varepsilon D_X)\tilde{\zeta}\big)\big((\mathcal{R}\omega_1^{-1/2})(k_0+\varepsilon D_X)(v_1)\big)\\
&\qquad+\mathcal{O}(\varepsilon^N)\\
&\quad= -\frac{\varepsilon_1^2}{2}\big((\mathcal{P}\mathcal{R})(k_0)|v_1|^2\big)\big(\varepsilon\mathcal{Q}^{(1,0)}\partial_X\tilde{\zeta} + \varepsilon^2\mathcal{Q}^{(1,1)}|D_X|\partial_x\tilde{\zeta} + \varepsilon^2\mathcal{Q}^{(2,0)}(|D_X|\partial_X\tilde{\zeta})\big)\\
&\qquad- \frac{\varepsilon_1^2}{2}\big((\mathcal{P}\omega_1^{1/2})(k_0)v_1\big)\big(\varepsilon\mathcal{Q}^{(1,0)}\partial_X\tilde{\zeta}\big)\big(\varepsilon(\mathcal{R}\omega_1^{-1/2})'(k_0)D_X\overline{v_1}\big)\\
&\qquad- \frac{\varepsilon_1^2}{2}\big((\mathcal{P}\mathcal{R})(k_0)|v_1|^2\big)\big(\varepsilon\mathcal{Q}^{(1,0)}\partial_X\tilde{\zeta} + \varepsilon^2\mathcal{Q}^{(1,1)}|D_X|\partial_x\tilde{\zeta} + \varepsilon^2\mathcal{Q}^{(2,0)}(|D_X|\partial_X\tilde{\zeta})\big)\\
&\qquad- \frac{\varepsilon_1^2}{2}\big(\varepsilon(\mathcal{P}\omega_1^{1/2})'(k_0)D_Xv_1\big)\big(\varepsilon\mathcal{Q}^{(1,0)}\partial_X\tilde{\zeta}\big)\big((\mathcal{R}\omega_1^{-1/2})(k_0)\overline{v_1}\big)\\
&\qquad+\mathcal{O}(\varepsilon^3\varepsilon_1^2),
\end{aligned}
\tag{6.43}
$$

which after simplification yields the result. □

6.3.3 *Cubic Terms of the Hamiltonian in Rescaled Variables*

Using Lemmas 6.8, 6.9 and 6.10 and the above equations, we calculate the contributions of each R_j to the cubic part of the Hamiltonian truncated to order $\mathcal{O}(\varepsilon^4)$.

Proposition 6.11. The cubic terms in the Hamiltonian from R_1 simplify to

$$
\begin{aligned}
R_1 = -\frac{1}{4}\frac{\sqrt{\rho-\rho_1}}{\sqrt{g}}\int_{\mathbb{R}}\Big[&2\varepsilon\varepsilon_1^2(b^+)^{(0)}(\mathcal{B}_1^2\omega_1^{-1})(k_0)\tilde{\mu}|v_1|^2\\
&+2\varepsilon^2\varepsilon_1^2(b^+)^{(1)}(\mathcal{B}_1^2\omega_1^{-1})(k_0)\big(|D_X|\tilde{\mu}\big)|v_1|^2\\
&+\varepsilon^2\varepsilon_1^2(b^+)^{(0)}(\mathcal{B}_1^2\omega_1^{-1})'(k_0)\tilde{\mu}\big[(D_Xv_1)\overline{v_1}+v_1(\overline{Dv_1})\big]\Big]\frac{dX}{\varepsilon}\\
&+\mathcal{O}(\varepsilon^4)
\end{aligned}
\tag{6.44}
$$

in rescaled variables.

Proof. From Equation (6.32), we have

$$\begin{aligned} R_1 &= -\frac{\rho-\rho_1}{2\sqrt{g(\rho-\rho_1)}}\int_{\mathbb{R}}(b^+\mu - b^-\mu_1)\left(\mathcal{A}_1\zeta - \mathcal{B}_1\zeta_1\right)^2\,dx \\ &= -\frac{\rho-\rho_1}{2\sqrt{g(\rho-\rho_1)}}\int_{\mathbb{R}}\left[(b^+\mu)(\mathcal{A}_1\zeta)^2 + (b^+\mu)(\mathcal{B}_1\zeta_1)^2\right. \\ &\quad \left.+\,2(b^-\mu_1)(\mathcal{A}_1\zeta)(\mathcal{B}_1\zeta_1)\right]dx + \mathcal{O}(\varepsilon^N), \end{aligned} \tag{6.45}$$

Using the relation, $(G_{11}^{(0)})^2 - (G_{12}^{(0)})^2 = (G^{(0)})^2$, we simplify the symbols $\mathcal{A}_1$ and $\mathcal{B}_1$. The symbols for the first term simplify to

$$\mathcal{A}_1 = \frac{a^+\sqrt{g\rho_1}(G_{11}^{(0)} + G^{(0)})}{-G_{12}^{(0)}(\rho G_{11}^{(0)} + \rho_1 G^{(0)})}(G^{(0)})^2 \tag{6.46}$$

and

$$\mathcal{B}_1 = \frac{a^+\sqrt{g}(\rho_1 G^{(0)} + (2\rho_1-\rho)G_{11}^{(0)})}{\sqrt{\rho-\rho_1}B_0}G^{(0)}. \tag{6.47}$$

Also, define $\mathcal{A}_1\zeta =: \mathcal{A}_1^{(2)}|D_x|^2\zeta = \varepsilon^2\mathcal{A}_1^{(2)}D_X^2\tilde{\zeta}$, where

$$\mathcal{A}_1^{(2)} = \frac{a^+\sqrt{g\rho_1}(G_{11}^{(0)} + G^{(0)})}{-G_{12}^{(0)}(\rho G_{11}^{(0)} + \rho_1 G^{(0)})}. \tag{6.48}$$

Since $\mathcal{A}_1^{(2)}$ and b^+ are even, we apply Proposition 6.8 for $\mathcal{P} = b^+$, $\mathcal{Q}^{(1)} = 0$ and $\mathcal{Q}^{(2)} = \mathcal{A}_1^{(2)}$ and deduce that

$$(b^+\mu)(\mathcal{A}_1^{(2)}D_x^2\zeta)^2 \sim \mathcal{O}(\varepsilon^5). \tag{6.49}$$

Next, applying Lemma 6.9, we expand the first cross-term where $\mathcal{B}_1$ is even

$$\begin{aligned} &(b^+\mu)(\mathcal{B}_1\zeta_1)^2 \\ &\quad= \varepsilon\varepsilon_1^2(b^+)^{(0)}(\mathcal{B}_1^2\omega_1^{-1})(k_0)\tilde{\mu}|v_1|^2 \\ &\quad+ \varepsilon^2\varepsilon_1^2(b^+)^{(1)}\left(|D_X|\tilde{\mu}\right)(\mathcal{B}_1^2\omega_1^{-1})(k_0)|v_1|^2 \\ &\quad+ \frac{\varepsilon^2\varepsilon_1^2}{2}(b^+)^{(0)}\tilde{\mu}(\mathcal{B}_1^2\omega_1^{-1})'(k_0)\left[(D_X v_1)\overline{v_1} + v_1(\overline{D_X v_1})\right] + \mathcal{O}(\varepsilon^3\varepsilon_1^2). \end{aligned} \tag{6.50}$$

To expand the last term in the first integrand, first observe that $\mathcal{B}_1$ is even – not odd – and so Proposition 6.10 is not applicable. Rather, applying Lemma 6.3 and noting $\mathcal{Q}^{(1)} = \mathcal{A}_1^{(1)} = 0$, the cross-term is

$$\begin{aligned}
&(b^-\mu_1)(\mathcal{A}_1\zeta)(\mathcal{B}_1\zeta_1)\\
&= -\frac{\varepsilon_1^2}{2i}(b^-\omega_1^{1/2})(D_x)(v_1e^{ik_0x}+\text{c.c.})(\mathcal{A}_1^{(2)}D_x^2\zeta)(\mathcal{B}_1\omega_1^{-1/2})(D_x)(v_1e^{ik_0x}-\text{c.c.})\\
&= +\frac{\varepsilon_1^2}{2i}\big(\varepsilon^2\mathcal{A}_1^{(2)}(\varepsilon D_X)D_X^2\tilde{\zeta}\big)\big((b^-\omega_1^{1/2})(k_0+\varepsilon D_X)v_1\big)\big((\mathcal{B}_1\omega_1^{-1/2})(-k_0+\varepsilon D_X)\overline{v_1}\big)\\
&\quad -\frac{\varepsilon_1^2}{2i}\big(\varepsilon^2\mathcal{A}_1^{(2)}(\varepsilon D_X)D_X^2\tilde{\zeta}\big)\big((b^-\omega_1^{1/2})(-k_0+\varepsilon D_X)\overline{v_1}\big)\big((\mathcal{B}_1\omega_1^{-1/2})(k_0+\varepsilon D_X)v_1\big)\\
&\quad +\mathcal{O}(\varepsilon_1^2\varepsilon^N)\\
&= \frac{\varepsilon_1^2\varepsilon^2}{2i}\big(\mathcal{A}_1^{(2)}(\varepsilon D_X)D_X^2\tilde{\zeta}\big)\big[(b^-\omega_1^{1/2})(k_0)(\mathcal{B}_1\omega_1^{-1/2})(-k_0)\\
&\quad -(b^-\omega_1^{1/2})(-k_0)(\mathcal{B}_1\omega_1^{-1/2})(-k_0)\big]+\mathcal{O}(\varepsilon^3\varepsilon_1^2)\lesssim\mathcal{O}(\varepsilon^3\varepsilon_1^2).
\end{aligned}\tag{6.51}$$

Combining terms, the proposition directly follows. □

Proposition 6.12. The cubic terms in the Hamiltonian from R_2 simplify to

$$\begin{aligned}
R_2 = \frac{1}{4}\frac{\sqrt{\rho_1}}{\sqrt{g}}\int_{\mathbb{R}}&\Big[2\varepsilon\varepsilon_1^2(a^+)^{(0)}(\mathcal{B}_2^2\omega_1^{-1})(k_0)\tilde{\mu}|v_1|^2\\
&+2\varepsilon^2\varepsilon_1^2(a^+)^{(1)}(\mathcal{B}_2^2\omega_1^{-1})(k_0)\big(|D_X|\tilde{\mu}\big)|v_1|^2\\
&+\varepsilon^2\varepsilon_1^2(a^+)^{(0)}(\mathcal{B}_2^2\omega_1^{-1})'(k_0)\tilde{\mu}\big[(Dv_1)\overline{v_1}+(\overline{Dv_1})v_1\big]\Big]\frac{dX}{\varepsilon}\\
&+\mathcal{O}(\varepsilon^4)
\end{aligned}\tag{6.52}$$

in rescaled variables.

Proof. The proof is similar to that of Proposition 6.11. Again we reduce the symbols $\mathcal{A}_2$ and $\mathcal{B}_2$ to

$$\begin{cases}
\mathcal{A}_2 &= \frac{a^+g(\rho-\rho_1)}{\sqrt{g\rho_1}(\rho G_{11}^{(0)}+\rho_1 G^{(0)})}(G^{(0)})^2\\
\mathcal{B}_2 &= \frac{a^+g\big((G_{11}^{(0)}+G^{(0)})(\rho G^{(0)}+\rho_1 G_{11}^{(0)})-(\rho-\rho_1)G_{12}^{(0)}\big)}{\sqrt{g(\rho-\rho_1)}(\rho G_{11}^{(0)}+\rho_1 G^{(0)})}G^{(0)}.
\end{cases}\tag{6.53}$$

and define $\mathcal{A}_2\zeta =: \mathcal{A}_2^{(2)}D_x^2\zeta = \varepsilon^2\mathcal{A}_2^{(2)}D_X^2\tilde{\zeta}$, where

$$\mathcal{A}_2^{(2)} := \frac{a^+g(\rho-\rho_1)}{\sqrt{g\rho_1}(\rho G_{11}^{(0)}+\rho_1 G^{(0)})}.\tag{6.54}$$

A direct application of Lemmas 6.8 and 6.9 informs us that

$$(a^+\mu)(\mathcal{A}_2\zeta)^2 \sim \mathcal{O}(\varepsilon^5), \tag{6.55}$$

$$\begin{aligned}(a^+\mu)(\mathcal{B}_2\zeta_1)^2 = {} & \varepsilon\varepsilon_1^2(a^+)^{(0)}\widetilde{\mu}(\mathcal{B}_2^2\omega_1^{-1})(k_0)|v_1|^2 \\ & + \varepsilon^2\varepsilon_1^2\big((a^+)^{(1)}|D_X|\widetilde{\mu}\big)(\mathcal{B}_2^2\omega_1^{-1})(k_0)|v_1|^2 \\ & + \frac{\varepsilon^2\varepsilon_1^2}{2}(a^+)^{(0)}\widetilde{\mu}(\mathcal{B}_2^2\omega_1^{-1})'(k_0)\left[(D_Xv_1)\overline{v_1} + v_1(\overline{D_Xv_1})\right] \\ & + \mathcal{O}(\varepsilon^3\varepsilon_1^2),\end{aligned} \tag{6.56}$$

and lastly, similar to our calculation in R_1, we have

$$(a^-\mu_1)(\mathcal{A}_2\zeta)(\mathcal{B}_2\zeta_1) \sim \mathcal{O}(\varepsilon^3\varepsilon_1^2). \tag{6.57}$$

Reading R_2 from Equation (6.32) and combining terms, we get the desired result. □

Proposition 6.13. The cubic terms in the Hamiltonian from R_3 reduce to

$$\begin{aligned}R_3 = \frac{\rho}{4\sqrt{g(\rho-\rho_1)}}\int_{\mathbb{R}} \Big[& 2\varepsilon\varepsilon_1^2(b^+)^{(0)}(\mathcal{B}_3^2\omega_1^{-1})(k_0)\widetilde{\mu}|v_1|^2 \\ & + 2\varepsilon^2\varepsilon_1^2(b^+)^{(1)}(\mathcal{B}_3^2\omega_1^{-1})(k_0)\big(|D_X|\widetilde{\mu}\big)|v_1|^2 \\ & + \varepsilon^2\varepsilon_1^2(b^+)^{(0)}(\mathcal{B}_3^2\omega_1^{-1})'(k_0)\widetilde{\mu}\left[(D_Xv_1)\overline{v_1} + v_1(\overline{D_Xv_1})\right] \\ & + 4\varepsilon^2\varepsilon_1^2\mathcal{A}_3^{(2,0)}(b^-\mathcal{B}_3)(k_0)(|D_X|\partial_X\widetilde{\zeta})|v_1|^2\Big]\,\frac{dX}{\varepsilon} + \mathcal{O}(\varepsilon^4)\end{aligned} \tag{6.58}$$

in rescaled coordinates.

Proof. We reduce the symbols $\mathcal{A}_3$ and $\mathcal{B}_3$ to

$$\begin{cases} \mathcal{A}_3 & = \frac{a^+\sqrt{g\rho_1}(G_{11}^{(0)}+G^{(0)})}{-G_{12}^{(0)}(\rho G_{11}^{(0)}+\rho_1 G^{(0)})} DG^{(0)} \\ \mathcal{B}_3 & = \frac{a^+\sqrt{g}(\rho_1 G^{(0)}+(2\rho_1-\rho)G_{11}^{(0)})}{\sqrt{\rho-\rho_1}B_0} D \end{cases} \tag{6.59}$$

and define $\mathcal{A}_3\zeta =: \mathcal{A}_3^{(2)}|D_x|D_x\zeta = \varepsilon^2\mathcal{A}_3^{(2)}|D_X|D_X\widetilde{\zeta}$, where

$$\mathcal{A}_3^{(2)} = \frac{a^+\sqrt{g\rho_1}(G_{11}^{(0)}+G^{(0)})}{-G_{12}^{(0)}(\rho G_{11}^{(0)}+\rho_1 G^{(0)})}. \tag{6.60}$$

Again, applying Propositions 6.8, 6.9 and 6.10, we get

$$(b^+\mu)(\mathcal{A}_3\zeta)^2 \sim \mathcal{O}(\varepsilon^5) \tag{6.61}$$

as well as the two cross-terms

$$\begin{aligned}(b^+\mu)(\mathcal{B}_3\zeta_1)^2 = &-\varepsilon\varepsilon_1^2(b^+)^{(0)}(\mathcal{B}_3^2\omega_1^{-1})(k_0)\widetilde{\mu}|v_1|^2 \\ &-\varepsilon^2\varepsilon_1^2(b^+)^{(1)}\big(|D_X|\widetilde{\mu}\big)(\mathcal{B}_3^2\omega_1^{-1})(k_0)|v_1|^2 \\ &-\frac{\varepsilon^2\varepsilon_1^2}{2}(b^+)^{(0)}(\mathcal{B}_3^2\omega_1^{-1})'(k_0)\widetilde{\mu}\big[(D_Xv_1)\overline{v_1}+v_1(\overline{D_Xv_1})\big] \\ &+\mathcal{O}(\varepsilon^3\varepsilon_1^2)\end{aligned} \tag{6.62}$$

and

$$\begin{aligned}&(b^-\mu_1)(\mathcal{A}_3\zeta)(\mathcal{B}_3\zeta_1) \\ &\quad=-\varepsilon^2\varepsilon_1^2\big(\mathcal{A}_3^{(2,0)}|D_X|\partial_X\widetilde{\zeta}\big)(b^-\mathcal{B}_3)(k_0)|v_1|^2+\mathcal{O}(\varepsilon^3\varepsilon_1^2).\end{aligned} \tag{6.63}$$

Combining terms, the result directly follows. □

Proposition 6.14. The cubic terms in the Hamiltonian from R_4 simplify to

$$\begin{aligned}R_4 = \frac{\rho_1}{4\sqrt{g(\rho-\rho_1)}}\int_{\mathbb{R}}\Big[&2\varepsilon^3(b^+)^{(0)}(A_4^{(1,0)})^2\widetilde{\mu}(D_X\widetilde{\zeta})^2 \\ &+2\varepsilon^4(b^+)^{(1)}(\mathcal{A}_4^{(1,0)})^2\big(|D_X|\widetilde{\mu}\big)(D_X\widetilde{\zeta})^2 \\ &+4\varepsilon^4(b^+)^{(0)}\mathcal{A}_4^{(1,0)}\mathcal{A}_4^{(1,1)}\widetilde{\mu}\big(|D_X|D_X\widetilde{\zeta}\big)(D_X\widetilde{\zeta}) \\ &-2\varepsilon\varepsilon_1^2(b^+)^{(0)}(\mathcal{B}_4^2\omega_1^{-1})(k_0)\widetilde{\mu}|v_1|^2-2\varepsilon^2\varepsilon_1^2(b^+)^{(1)}(\mathcal{B}_4^2\omega_1^{-1})(k_0)\big(|D_X|\widetilde{\mu}\big)|v_1|^2 \\ &-4\varepsilon\varepsilon_1^2\mathcal{A}_4^{(1,0)}(b^-\mathcal{B}_4)(k_0)(\partial_X\widetilde{\zeta})|v_1|^2-4\varepsilon^2\varepsilon_1^2\mathcal{A}_4^{(1,1)}(b^-\mathcal{B}_4)(k_0)\big(|D_X|\partial_X\widetilde{\zeta}\big)|v_1|^2 \\ &-\varepsilon^2\varepsilon_1^2(b^+)^{(0)}(\mathcal{B}_4^2\omega^{-1})'(k_0)\widetilde{\mu}[(Dv_1)\overline{v_1}+v_1(\overline{Dv_1})] \\ &+4\varepsilon^4(b^+)^{(0)}\mathcal{A}_4^{(1,0)}\mathcal{A}_4^{(2,0)}\widetilde{\mu}(D_X\widetilde{\zeta})(|D_X|D_X\widetilde{\zeta}) \\ &-4\varepsilon^2\varepsilon_1^2\mathcal{A}_4^{(2,0)}(b^-\mathcal{B}_4)(k_0)(|D_X|\partial_X\widetilde{\zeta})|v_1|^2 \\ &-2\varepsilon^2\varepsilon_1^2\mathcal{A}_4^{(1,0)}(b^-\mathcal{B}_4)'(k_0)(\partial_X\widetilde{\zeta})\big[(Dv_1)\overline{v_1}+v_1(\overline{Dv_1})\big]\Big]\frac{dX}{\varepsilon} \\ &+\mathcal{O}(\varepsilon^4)\end{aligned} \tag{6.64}$$

in rescaled coordinates.

Proof. We define

$$\mathcal{A}_4 := \mathcal{A}_4^{(1)}D_x+\mathcal{A}_4^{(2)}|D_x|D_x \tag{6.65}$$

and read the symbols

$$\mathcal{A}_4^{(1)} := -\frac{\rho}{\rho_1}a^+\sqrt{g\rho_1}B_0^{-1}G_{12}^{(0)} \tag{6.66}$$

and

$$\mathcal{A}_4^{(2)} := b^+\sqrt{g(\rho-\rho_1)}B_0^{-1}. \tag{6.67}$$

Applying Proposition 6.8 with $\mathcal{P} = b^+$, $\mathcal{Q}^{(1)} = \mathcal{A}_4^{(1)}$ and $\mathcal{Q}^{(2)} = \mathcal{A}_4^{(2)}$, we compute

$$\begin{aligned}
&(b^+\mu)(\mathcal{A}_4\zeta)^2 \\
&\quad = \varepsilon^3 (b^+)^{(0)} (A_4^{(1,0)})^2 \widetilde{\mu}(D_X\widetilde{\zeta})^2 \\
&\qquad + \varepsilon^4 (\mathcal{A}_4^{(1,0)})^2 \big((b^+)^{(1)}|D_X|\widetilde{\mu}\big)(D_X\widetilde{\zeta})^2 \\
&\qquad + 2\varepsilon^4 (b^+)^{(0)} \mathcal{A}_4^{(1,0)} \mathcal{A}_4^{(1,1)} \widetilde{\mu}(D_X\widetilde{\zeta})\big(|D_X|D_X\widetilde{\zeta}\big) \\
&\qquad + 2\varepsilon^4 (b^+)^{(0)} \mathcal{A}_4^{(1,0)} \mathcal{A}_4^{(2,0)} \widetilde{\mu}(D_X\widetilde{\zeta})(|D_X|D_X\widetilde{\zeta}) + \mathcal{O}(\varepsilon^5).
\end{aligned} \tag{6.68}$$

Next, we apply Proposition 6.9 to get

$$\begin{aligned}
&(b^+\mu)(\mathcal{B}_4\zeta_1)^2 \\
&\quad = -\varepsilon\varepsilon_1^2 (b^+)^{(0)} \widetilde{\mu}(\mathcal{B}_4^2\omega_1^{-1})(k_0)|v_1|^2 \\
&\qquad - \varepsilon^2\varepsilon_1^2 (b^+)^{(1)} (\mathcal{B}_4^2\omega_1^{-1})(k_0)\big(|D_X|\widetilde{\mu}\big)|v_1|^2 \\
&\qquad - \frac{\varepsilon\varepsilon_1^2}{2}(b^+)^{(0)} \widetilde{\mu}(\mathcal{B}_4^2\omega^{-1})'(k_0)[(D_Xv_1)\overline{v_1} + v_1(\overline{D_Xv_1})] + \mathcal{O}(\varepsilon^3\varepsilon_1^2),
\end{aligned} \tag{6.69}$$

and lastly, by Proposition 6.10, we calculate

$$\begin{aligned}
&(b^-\mu_1)(\mathcal{A}_4\zeta)(\mathcal{B}_4\zeta_1) \\
&\quad = -\varepsilon\varepsilon_1^2 \mathcal{A}_4^{(1,0)} (b^-\mathcal{B}_4)(k_0)(\partial_X\widetilde{\zeta})|v_1|^2 \\
&\qquad - \varepsilon^2\varepsilon_1^2 \mathcal{A}_4^{(1,1)} (b^-\mathcal{B}_4)(k_0)\big(|D_X|\partial_X\widetilde{\zeta}\big)|v_1|^2 \\
&\qquad - \varepsilon^2\varepsilon_1^2 \mathcal{A}_4^{(2,0)} |D_X|\partial_X\widetilde{\zeta})(b^-\mathcal{B}_4)(k_0)|v_1|^2 \\
&\qquad - \frac{\varepsilon^2\varepsilon_1^2}{2} \mathcal{A}_4^{(1,0)} (b^-\mathcal{B}_4)'(k_0)(\partial_X\widetilde{\zeta})\big[(D_Xv_1)\overline{v_1} + v_1(\overline{D_Xv_1})\big] \\
&\qquad + \mathcal{O}(\varepsilon^3\varepsilon_1^2),
\end{aligned} \tag{6.70}$$

which completes the proof. □

Proposition 6.15. The cubic terms in the Hamiltonian from R_5 simplify to

$$\begin{aligned}
R_5 = \frac{1}{4\rho_1\sqrt{g\rho_1}} \int_{\mathbb{R}} \Big[& 2\varepsilon^3 a^+(0)(\mathcal{A}_5^{(1,0)})^2 \widetilde{\mu}(D_X\widetilde{\zeta})^2 \\
& + 2\varepsilon^4 (a^+)^{(1)} (\mathcal{A}_5^{(1,0)})^2 \big(|D_X|\widetilde{\mu}\big)(D_X\widetilde{\zeta})^2 \\
& + 4\varepsilon^4 (a^+)^{(0)} \mathcal{A}_5^{(1,0)} \mathcal{A}_5^{(1,1)} \big(|D_X|D_X\widetilde{\zeta}\big)\widetilde{\mu}(D_X\widetilde{\zeta}) \\
& - 2\varepsilon\varepsilon_1^2 (a^+)^{(0)} (\mathcal{B}_5^2\omega_1^{-1})(k_0)\widetilde{\mu}|v_1|^2 - 2\varepsilon^2\varepsilon_1^2 (a^+)^{(1)} (\mathcal{B}_5^2\omega_1^{-1})(k_0)\big(|D_X|\widetilde{\mu}\big)|v_1|^2 \\
& - 4\varepsilon\varepsilon_1^2 \mathcal{A}_5^{(1,0)} (a^-\mathcal{B}_5)(k_0)(D_X\widetilde{\zeta})|v_1|^2 - 4\varepsilon^2\varepsilon_1^2 \mathcal{A}_5^{(1,1)} (a^-\mathcal{B}_5)(k_0)\big(|D_X|\partial_X\widetilde{\zeta}\big)|v_1|^2 \\
& - \varepsilon^2\varepsilon_1^2 (a^+)^{(0)} (\mathcal{B}_5\omega_1^{-1})'(k_0)\widetilde{\mu}\big[(D_X v_1)\overline{v_1} + v_1(\overline{D_X v_1})\big] \\
& - 2\varepsilon^2\varepsilon_1^2 \mathcal{A}_5^{(1,0)} (a^-\mathcal{B}_5)'(k_0)(\partial_X\widetilde{\zeta})\big[(D_X v_1)\overline{v_1} + v_1(\overline{D_X v_1})\big]\Big] \frac{dX}{\varepsilon} \\
& + \mathcal{O}(\varepsilon^4)
\end{aligned} \tag{6.71}$$

in rescaled variables.

Proof. First define $\mathcal{A}_5 := \mathcal{A}_5^{(1)} D_X + \mathcal{A}_5^{(2)}|D_X|D_X$, where

$$\mathcal{A}_5^{(1)} = -\sqrt{g\rho_1}a^+(D) \tag{6.72}$$

and $\mathcal{A}_5^{(2)} = 0$. Then, applying Lemmas 6.8, 6.9 and 6.10, we simplify the terms

$$\begin{aligned}
(a^+\mu)(\mathcal{A}_5\zeta)^2 = {} & \varepsilon^3 (a^+)^{(0)} (\mathcal{A}_5^{(1)}(0))^2 \widetilde{\mu}(D_X\widetilde{\zeta})^2 \\
& + \varepsilon^4 (a^+)^{(1)} (\mathcal{A}_5^{(1,0)})^2 \big(|D_X|\widetilde{\mu}\big)(D_X\widetilde{\zeta})^2 \\
& + 2\varepsilon^4 (a^+)^{(0)} \mathcal{A}_5^{(1,0)} \mathcal{A}_5^{(1,1)} \widetilde{\mu}(D_X\widetilde{\zeta})\big(|D_X|D_X\widetilde{\zeta}\big) \\
& + \mathcal{O}(\varepsilon^5),
\end{aligned} \tag{6.73}$$

$$\begin{aligned}
& (a^+\mu)(\mathcal{B}_5\zeta_1)^2 \\
& \quad = -\varepsilon\varepsilon_1^2 (a^+)^{(0)} \widetilde{\mu}(\mathcal{B}_5^2\omega_1^{-1})(k_0)|v_1|^2 \\
& \qquad - \varepsilon^2\varepsilon_1^2 (a^+)^{(1)} (\mathcal{B}_5^2\omega_1^{-1})(k_0)\big(|D_X|\widetilde{\mu}\big)|v_1|^2 \\
& \qquad - \frac{\varepsilon^2\varepsilon_1^2}{2} (a^+)^{(0)} (\mathcal{B}_5^2\omega_1^{-1})'(k_0)\widetilde{\mu}\big[(D_X v_1)\overline{v_1} + v_1(\overline{D_X v_1})\big] + \mathcal{O}(\varepsilon^3\varepsilon_1^2),
\end{aligned} \tag{6.74}$$

and lastly

$$
\begin{aligned}
&(a^-\mu_1)(\mathcal{A}_5\zeta)(\mathcal{B}_5\zeta_1)\\
&\quad = -\varepsilon\varepsilon_1^2\mathcal{A}_5^{(1,0)}(a^-\mathcal{B}_5)(k_0)(D_X\widetilde{\zeta})|v_1|^2\\
&\qquad - \varepsilon^2\varepsilon_1^2\mathcal{A}_5^{(1,1)}(a^-\mathcal{B}_5)(k_0)\big(|D_X|\partial_X\widetilde{\zeta}\big)|v_1|^2\\
&\qquad - \frac{\varepsilon^2\varepsilon_1^2}{2}\mathcal{A}_5^{(1,0)}(a^-\mathcal{B}_5)'(k_0)(\partial_X\widetilde{\zeta})\big[(D_Xv_1)\overline{v_1} + v_1(\overline{D_Xv_1})\big]\\
&\qquad + \mathcal{O}(\varepsilon^3\varepsilon_1^2),
\end{aligned}
\tag{6.75}
$$

which completes the proof. □

Adding all of these contributions from each R_j term together, we find the cubic part of the Hamiltonian truncated at order $\mathcal{O}(\varepsilon^4)$ in rescaled variables in the Benjamin-Ono scaling regime.

Proof of Proposition 6.6. This follows directly from Propositions 6.11, 6.12, 6.13, 6.14 and 6.15.

We group the relevant terms of the cubic terms of the Hamiltonian $H^{(3)}$ from Equation (6.26), writing the coefficients explicitly

$$
\begin{aligned}
\kappa &= \frac{\rho_1}{2\sqrt{g(\rho-\rho_1)}}(b^+)^{(0)}(\mathcal{A}_4^{(1,0)})^2 + \frac{1}{2\rho_1\sqrt{g\rho_1}}(a^+)^{(0)}(\mathcal{A}_5^{(1,0)})^2\\
&= \frac{\sqrt{g(1-\gamma)}}{2\sqrt{\rho_1}},
\end{aligned}
\tag{6.76}
$$

$$
\begin{aligned}
\kappa_1 = &-\frac{1}{2}\sqrt{\frac{\rho-\rho_1}{g}}(b^+)^{(0)}(\mathcal{B}_1^2\omega_1^{-1})(k_0) + \frac{1}{2}\sqrt{\frac{\rho_1}{g}}(a^+)^{(0)}(\mathcal{B}_2^2\omega_1^{-1})(k_0)\\
&+ \frac{\rho}{2\sqrt{g(\rho-\rho_1)}}(b^+)^{(0)}(\mathcal{B}_3^2\omega_1^{-1})(k_0) - \frac{\rho_1}{2\sqrt{g(\rho-\rho_1)}}(b^+)^{(0)}(\mathcal{B}_4^2\omega_1^{-1})(k_0)\\
&- \frac{1}{2\rho_1\sqrt{g\rho_1}}(a^+)^{(0)}(\mathcal{B}_5^2\omega_1^{-1})(k_0),
\end{aligned}
\tag{6.77}
$$

$$
\begin{aligned}
\kappa_2 &= -\frac{\rho_1}{\sqrt{g(\rho-\rho_1)}}A_4^{(1)}(0)(b^-\mathcal{B}_4)(k_0) - \frac{1}{\rho_1\sqrt{g\rho_1}}A_5^{(1)}(0)(a^-\mathcal{B}_5)(k_0)\\
&= \sqrt{\gamma}(b^-\mathcal{B}_4)(k_0) + \frac{\sqrt{1-\gamma}}{\rho_1}(a^-\mathcal{B}_5)(k_0),
\end{aligned}
\tag{6.78}
$$

$$\begin{aligned}
\kappa_3 &= \frac{\rho_1}{\sqrt{g(\rho-\rho_1)}}(b^+)^{(0)}\mathcal{A}_4^{(1,0)}(\mathcal{A}_4^{(2,0)}+\mathcal{A}_4^{(1,1)}) + \frac{1}{\rho_1\sqrt{g\rho_1}}(a^+)^{(0)}\mathcal{A}_5^{(1,0)}\mathcal{A}_5^{(1,1)} \\
&= \frac{\rho_1}{\sqrt{g(\rho-\rho_1)}}\sqrt{\gamma}\frac{\sqrt{g(1-\gamma)}}{\sqrt{\rho_1}}\left(-\frac{\sqrt{g(1-\gamma)}}{\sqrt{\rho_1}}\gamma h_1\right) \\
&\quad + \frac{1}{\rho_1\sqrt{g\rho_1}}\sqrt{1-\gamma}\left(-\sqrt{g\rho_1(1-\gamma)}\right)\left(\sqrt{g\rho_1(1-\gamma)}\gamma h_1\right) \\
&= \frac{-\sqrt{g(1-\gamma)}\gamma h_1}{\sqrt{\rho_1}},
\end{aligned} \tag{6.79}$$

$$\begin{aligned}
\kappa_4 &= -\frac{\sqrt{\rho-\rho_1}}{4\sqrt{g}}(b^+)^{(0)}(\mathcal{B}_1^2\omega_1^{-1})'(k_0) + \frac{\sqrt{\rho_1}}{4\sqrt{g}}(a^+)^{(0)}(\mathcal{B}_2^2\omega_1^{-1})'(k_0) \\
&\quad + \frac{\rho}{\sqrt{4g(\rho-\rho_1)}}(b^+)^{(0)}(\mathcal{B}_3^2\omega_1^{-1})'(k_0) - \frac{\rho_1}{4\sqrt{g(\rho-\rho_1)}}(b^+)^{(0)}(\mathcal{B}_4^2\omega^{-1})'(k_0) \\
&\quad - \frac{1}{4\rho_1\sqrt{g\rho_1}}(a^+)^{(0)}(\mathcal{B}_5^2\omega_1^{-1})'(k_0),
\end{aligned} \tag{6.80}$$

$$\kappa_5 = -\frac{\rho_1}{2\sqrt{g(\rho-\rho_1)}}\mathcal{A}_4^{(1,0)}(b^-\mathcal{B}_4)'(k_0) - \frac{1}{2\rho_1\sqrt{g\rho_1}}\mathcal{A}_5^{(1,0)}(a^-\mathcal{B}_5)'(k_0), \tag{6.81}$$

$$\begin{aligned}
\kappa_6 &= \frac{-\sqrt{\rho-\rho_1}}{2\sqrt{g}}(b^+)^{(1)}(\mathcal{B}_1^2\omega_1^{-1})(k_0) + \frac{\sqrt{\rho_1}}{2\sqrt{g}}(a^+)^{(1)}(\mathcal{B}_2^2\omega_1^{-1})(k_0) \\
&\quad + \frac{\rho}{2\sqrt{g(\rho-\rho_1)}}(b^+)^{(1)}(\mathcal{B}_3^2\omega_1^{-1})(k_0) - \frac{\rho_1}{2\sqrt{g(\rho-\rho_1)}}(b^+)^{(1)}(\mathcal{B}_4^2\omega_1^{-1})(k_0) \\
&\quad - \frac{1}{2\rho_1\sqrt{g\rho_1}}(a^+)^{(1)}(\mathcal{B}_5^2\omega_1^{-1})(k_0),
\end{aligned} \tag{6.82}$$

$$\begin{aligned}
\kappa_7 &= \frac{\rho}{\sqrt{g(\rho-\rho_1)}}\mathcal{A}_3^{(2,0)}(b^-\mathcal{B}_3)(k_0) - \frac{\rho_1}{\sqrt{g(\rho-\rho_1)}}(\mathcal{A}_4^{(2,0)}+\mathcal{A}_4^{(1,1)})(b^-\mathcal{B}_4)(k_0) \\
&\quad - \frac{1}{\rho_1\sqrt{g\rho_1}}\mathcal{A}_5^{(1,1)}(a^-\mathcal{B}_5)(k_0),
\end{aligned} \tag{6.83}$$

and

$$
\begin{aligned}
\kappa_8 &= \frac{\rho_1}{2\sqrt{g(\rho-\rho_1)}}(b^+)^{(1)}(\mathcal{A}_4^{(1,0)})^2 + \frac{1}{2\rho_1\sqrt{g\rho_1}}(a^+)^{(1)}(\mathcal{A}_5^{(1,0)})^2 \\
&= \frac{\rho_1}{2\sqrt{g(\rho-\rho_1)}}(\sqrt{\gamma}(1-\gamma)h_1)\left(\frac{g(1-\gamma)}{\rho_1}\right) \\
&\quad + \frac{1}{2\rho_1\sqrt{g\rho_1}}(-\gamma\sqrt{1-\gamma}h_1)\big(g\rho_1(1-\gamma)\big) \\
&= 0,
\end{aligned}
\tag{6.84}
$$

respectively, which completes the proof. □

7

THE COUPLED BENJAMIN-ONO AND SCHRÖDINGER SYSTEM

7.1 TRANSFORMATION THEORY

As a prelude to the derivation of the coupled system, we develop the necessary transformation theory. We start with Hamilton's equations from the water wave problem in normal coordinates from Lemma 4.8

$$\partial_t \begin{pmatrix} \mu \\ \zeta \\ \mu_1 \\ \zeta_1 \end{pmatrix} \equiv J\nabla H = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_\mu H \\ \delta_\zeta H \\ \delta_{\mu_1} H \\ \delta_{\zeta_1} H \end{pmatrix}. \tag{7.1}$$

We can also express the Hamiltonian both without and with scaled coordinates in the forms $H[\mu,\zeta,\mu_1,\zeta_1]_x$ and $\widetilde{H}[\widetilde{\mu},\widetilde{\zeta},v_1,\overline{v_1}]_X$, where we integrate with respect to x and long-wave variable $X = \varepsilon x$, respectively. Now we prove two short lemmas with respect to the symplectic structure of the Hamiltonian.

Lemma 7.1. *Hamilton's equations and the symplectic map for $(\widetilde{\mu},\widetilde{\zeta})$ are given by*

$$\partial_t \begin{pmatrix} \widetilde{\mu} \\ \widetilde{\zeta} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_{\widetilde{\mu}} \widetilde{H} \\ \delta_{\widetilde{\zeta}} \widetilde{H} \end{pmatrix}. \tag{7.2}$$

Proof. First we let $(f,g)_x = \int_{\mathbb{R}} fg\,dx$ and $(f,g)_X = \int_{\mathbb{R}} fg\,dX$. Next, under the Benjamin-Ono scaling, $\widetilde{\zeta}(X) = \zeta(x)$ and $\widetilde{v}(X) = v(x)$, we verify

$$\begin{aligned}
(\delta_\zeta H, v)_x &= \lim_{\delta\to 0^+} \frac{1}{\delta}(H[\mu,\zeta+\delta v,\mu_1,\zeta_1]_x - H[\mu,\zeta,\mu_1,\zeta_1]_x) \\
&= \lim_{\delta\to 0^+} \frac{1}{\delta}(\widetilde{H}[\widetilde{\mu},\widetilde{\zeta}+\delta\widetilde{v},v_1,\overline{v_1}]_X - \widetilde{H}[\widetilde{\mu},\widetilde{\zeta},v_1,\overline{v_1}]_X) \\
&= (\delta_{\widetilde{\zeta}}\widetilde{H},\widetilde{v})_X \\
&= \varepsilon(\delta_{\widetilde{\zeta}}\widetilde{H},v)_x,
\end{aligned} \tag{7.3}$$

whence $\delta_\zeta H = \varepsilon \delta_{\tilde{\zeta}} \widetilde{H}$. Under the modulational Ansatz, we calculate

$$\partial_t \widetilde{\mu} = \varepsilon^{-1} \partial_t \mu = \varepsilon^{-1} \delta_\zeta H = \delta_{\tilde{\zeta}} \widetilde{H}. \tag{7.4}$$

Similarly, we use the Benjamin-Ono scaling $\mu(x) = \varepsilon \widetilde{\mu}(X)$ and $v(x) = \varepsilon \widetilde{v}(X)$ to check

$$(\delta_\mu H, v)_x = (\delta_{\tilde{\mu}} \widetilde{H}, \widetilde{v})_X = (\delta_{\tilde{\mu}} \widetilde{H}, v)_x, \tag{7.5}$$

whence $\delta_\mu H = \delta_{\tilde{\mu}} \widetilde{H}$. Then, again under the modulational Ansatz, we calculate

$$\partial_t \widetilde{\zeta} = \partial_t \zeta = -\delta_\mu H = -\delta_{\tilde{\mu}} \widetilde{H}, \tag{7.6}$$

which completes the proof. □

Lemma 7.2. *Hamilton's equations and the symplectic map for* $(v_1, \overline{v_1})$ *are given by*

$$\partial_t \begin{pmatrix} v_1 \\ \overline{v_1} \end{pmatrix} = \begin{pmatrix} 0 & -i\varepsilon\varepsilon_1^{-2} \\ i\varepsilon\varepsilon_1^{-2} & 0 \end{pmatrix} \begin{pmatrix} \delta_{v_1} \widetilde{H} \\ \delta_{\overline{v_1}} \widetilde{H} \end{pmatrix}. \tag{7.7}$$

Proof. As in Lemma 7.1, we use the inverse of the Benjamin-Ono scaling from Equation (6.12),

$$\begin{cases} v_1(X,t) &= \frac{1}{\sqrt{2}\varepsilon_1} e^{-ik_0 x} \left(\frac{1}{\sqrt{\omega_1}} \mu_1 + i\sqrt{\omega_1} \zeta_1 \right) \\ \overline{v_1}(X,t) &= \frac{1}{\sqrt{2}\varepsilon_1} e^{ik_0 x} \left(\frac{1}{\sqrt{\omega_1}} \mu_1 - i\sqrt{\omega_1} \zeta_1 \right), \end{cases} \tag{7.8}$$

as well as the modulational scaling given by $\mu_1(x) = \widetilde{\mu_1}(X)$, $\zeta_1(x) = \widetilde{\zeta_1}(X)$ and $v(x) = \widetilde{v}(X)$ to check

$$\begin{cases} (\delta_{\mu_1} H, v)_x &= (\delta_{\widetilde{\mu_1}} \widetilde{H}, \widetilde{v})_X = \varepsilon(\delta_{\mu_1} H, v)_x \\ (\delta_{\zeta_1} H, v)_x &= (\delta_{\widetilde{\zeta_1}} \widetilde{H}, \widetilde{v})_X = \varepsilon(\delta_{\zeta_1} H, v)_x. \end{cases} \tag{7.9}$$

Finally, we perform the modulational Ansatz

$$
\begin{aligned}
\partial_t \begin{pmatrix} v_1 \\ \overline{v_1} \end{pmatrix} &= \begin{pmatrix} \frac{\partial v_1}{\partial \mu_1} & \frac{\partial v_1}{\partial \zeta_1} \\ \frac{\partial \overline{v_1}}{\partial \mu_1} & \frac{\partial \overline{v_1}}{\partial \zeta_1} \end{pmatrix} \partial_t \begin{pmatrix} \widetilde{\mu_1} \\ \widetilde{\zeta}_1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial v_1}{\partial \mu_1} & \frac{\partial v_1}{\partial \zeta_1} \\ \frac{\partial \overline{v_1}}{\partial \mu_1} & \frac{\partial \overline{v_1}}{\partial \zeta_1} \end{pmatrix} \begin{pmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{pmatrix} \begin{pmatrix} \delta_{\widetilde{\mu_1}} \widetilde{H} \\ \delta_{\widetilde{\zeta}_1} \widetilde{H} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial v_1}{\partial \mu_1} & \frac{\partial v_1}{\partial \zeta_1} \\ \frac{\partial \overline{v_1}}{\partial \mu_1} & \frac{\partial \overline{v_1}}{\partial \zeta_1} \end{pmatrix} \begin{pmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial v_1}{\partial \mu_1} & \frac{\partial \overline{v_1}}{\partial \mu_1} \\ \frac{\partial v_1}{\partial \zeta_1} & \frac{\partial \overline{v_1}}{\partial \zeta_1} \end{pmatrix} \begin{pmatrix} \delta_{v_1} \widetilde{H} \\ \delta_{\overline{v_1}} \widetilde{H} \end{pmatrix} \\
&= \begin{pmatrix} 0 & -i\varepsilon\varepsilon_1^{-2} \\ i\varepsilon\varepsilon_1^{-2} & 0 \end{pmatrix} \begin{pmatrix} \delta_{v_1} \widetilde{H} \\ \delta_{\overline{v_1}} \widetilde{H} \end{pmatrix},
\end{aligned}
\tag{7.10}
$$

which completes the proof. □

Finally, we derive Hamilton's equations of motion with respect to variables $\widetilde{\mu}$ and $\widetilde{u} := \partial_X \widetilde{\zeta}$, the latter of which measures the horizontal shear velocity.

Lemma 7.3. *Hamilton's equations and the symplectic map for transformed variables* $(\widetilde{\mu}, \widetilde{u} := \partial_X \widetilde{\zeta})$ *satisfy*

$$
\partial_t \begin{pmatrix} \widetilde{\mu} \\ \widetilde{u} \end{pmatrix} = \begin{pmatrix} 0 & -\partial_X \\ -\partial_X & 0 \end{pmatrix} \begin{pmatrix} \delta_{\widetilde{\mu}} \widetilde{H} \\ \delta_{\widetilde{u}} \widetilde{H} \end{pmatrix}. \tag{7.11}
$$

Proof. Using Lemma 7.2, we readily calculate

$$
\partial_t \widetilde{u} = \partial_X (\partial_t \widetilde{\zeta}) = -\partial_X (\delta_{\widetilde{\mu}} \widetilde{H}). \tag{7.12}
$$

Letting H denote the rescaled Hamiltonian in $\widetilde{\mu}$, $\widetilde{\zeta}$ and long-wave variable X, we use the scaling $\widetilde{v}(X) = \partial_X v(X)$ to check

$$
(\delta_{\widetilde{\zeta}} H, v)_X = (\delta_{\widetilde{u}} \widetilde{H}, \widetilde{v})_X = (\delta_{\widetilde{u}} \widetilde{H}, \partial_X v)_X = -(\partial_X \delta_{\widetilde{u}} \widetilde{H}, v)_X, \tag{7.13}
$$

whence $\delta_{\widetilde{\zeta}} H = -\partial_X \delta_{\widetilde{u}} \widetilde{H}$. Finally, we calculate

$$
\partial_t \widetilde{\mu} = \delta_{\widetilde{\zeta}} H = -\partial_X \delta_{\widetilde{u}} \widetilde{H}, \tag{7.14}
$$

which completes the proof. □

7.2 RESONANCE CONDITION

We define modified Hamiltonian

$$\widehat{H} = H - cI - \frac{\varepsilon_1^2}{\varepsilon}(\omega_1(k_0) - ck_0)M, \tag{7.15}$$

where the L^2 norm

$$M := \int_{\mathbb{R}} |v_1|^2 \, dX \tag{7.16}$$

is also a conserved quantity of the system. This can be readily verified by showing that the Poisson commutator $\{M, H - cI\} = 0$ and so the Hamiltonian dynamics for this new quantity are preserved. Next we calculate quantity

$$H^{(2)} - cI - \frac{\varepsilon_1^2}{\varepsilon}(\omega_1(k_0) - ck_0)M, \tag{7.17}$$

in which we temporarily omit the cubic contribution $H^{(3)}$, in the rescaled variables.

Lemma 7.4. *The quantity* $H^{(2)} - cI - \frac{\varepsilon_1^2}{\varepsilon}(\omega_1(k_0) - ck_0)M$ *is*

$$\begin{aligned} &H^{(2)} - cI - \frac{\varepsilon_1^2}{\varepsilon}(\omega_1(k_0) - ck_0)M \\ &\quad = \int_{\mathbb{R}} \Bigg[\varepsilon \frac{(\omega^2)^{(2)}}{4} \left[\frac{2\widetilde{\mu}^2}{(\omega^2)^{(2)}} \left(1 - \frac{2c^2}{(\omega^2)^{(2)}}\right) - \left(D_X \widetilde{\zeta} + \frac{2ic\widetilde{\mu}}{(\omega^2)^{(2)}}\right)^2 \right] \\ &\qquad + \varepsilon^2 \frac{(\omega^2)^{(3)}}{12} \widetilde{\zeta}(D_X^2 |D_X| \widetilde{\zeta}) + \varepsilon^3 \frac{(\omega^2)^{(4)}}{48} \widetilde{\zeta}(D_X^4 \widetilde{\zeta}) \\ &\qquad + \frac{\varepsilon_1^2}{2}(\omega_1'(k_0) - c)\left[(D_X v_1)\overline{v_1} + v_1(\overline{D_X v_1})\right] \\ &\qquad + \frac{\varepsilon\varepsilon_1^2}{2}\omega_1''(k_0)\overline{v_1}(D_X^2 v_1) \Bigg] \, dX + \mathcal{O}(\varepsilon^4) \end{aligned} \tag{7.18}$$

in the rescaled variables for the Benjamin-Ono and modulational regimes.

Proof. We add the two components from Lemmas 6.4 and 6.5

$$
\begin{aligned}
&H^{(2)} - cI - \frac{\varepsilon_1^2}{\varepsilon}(\omega_1(k_0) - ck_0)M \\
&\quad = \int_{\mathbb{R}} \Big[-\varepsilon \frac{(\omega^2)^{(2)}}{4}(D_X\widetilde{\zeta})^2 + \varepsilon^2 \frac{(\omega^2)^{(3)}}{12}\widetilde{\zeta}(D_X^2|D_X|\widetilde{\zeta}) + \varepsilon^3 \frac{(\omega^2)^{(4)}}{48}\widetilde{\zeta}(D_X^4\widetilde{\zeta}) \\
&\qquad + \varepsilon\widetilde{\mu}^2 + \frac{\varepsilon_1^2}{\varepsilon}\omega_1(k_0)|v_1|^2 + \varepsilon_1^2\omega_1'(k_0)\overline{v_1}(D_X v_1) + \varepsilon_1^2\varepsilon\frac{\omega_1''(k_0)}{2}\overline{v_1}(D_X^2 v_1) \\
&\qquad + ic\varepsilon\widetilde{\zeta}(D_X\widetilde{\mu}) - \frac{c\varepsilon_1^2}{\varepsilon}k_0|v_1|^2 - \frac{c\varepsilon_1^2}{2}\left[(D_X v_1)\overline{v_1} + v_1(\overline{D_X v_1})\right] \\
&\qquad - \frac{\varepsilon_1^2}{\varepsilon}(\omega_1(k_0) - c_0k_0)|v_1|^2 \Big] \, dX + \mathcal{O}(\varepsilon^4) \\
&\quad = \int_{\mathbb{R}} \Big[-\varepsilon \frac{(\omega^2)^{(2)}}{4}(D_X\widetilde{\zeta})^2 + \varepsilon^2 \frac{(\omega^2)^{(3)}}{12}\widetilde{\zeta}(D_X^2|D_X|\widetilde{\zeta}) + \varepsilon^3 \frac{(\omega^2)^{(4)}}{48}\widetilde{\zeta}(D_X^4\widetilde{\zeta}) \\
&\qquad + \varepsilon\widetilde{\mu}^2 - ic\varepsilon(D_X\widetilde{\zeta})\widetilde{\mu} + \frac{\varepsilon_1^2}{2}(\omega_1'(k_0) - c)\left[(D_X v_1)\overline{v_1} + v_1(\overline{D_X v_1})\right] \\
&\qquad + \frac{\varepsilon\varepsilon_1^2}{2}\omega_1''(k_0)\overline{v_1}(D_X^2 v_1) \Big] \, dX + \mathcal{O}(\varepsilon^4),
\end{aligned}
\tag{7.19}
$$

which, after some algebraic manipulation, yields the result. □

The equation for this modified Hamiltonian $\widehat{H}$ can be simplified with the choice of

$$
c = c_0 = \sqrt{\frac{(\omega^2)^{(2)}}{2}} \tag{7.20}
$$

and wavenumber k_0 such that

$$
\omega_1'(k_0) = c_0, \tag{7.21}
$$

which can be interpreted as the resonant condition between the internal and surface modes that occurs when both the group velocity $\omega_1'(k_0)$ and the phase velocity c_0 coincide.

Lemma 7.5. *The constants c_0 and k_0 satisfying Equations (7.20) and (7.21) are*

$$
c_0 = \sqrt{g(1-\gamma)h_1}; \quad k_0 = \frac{1}{4h_1(1-\gamma)}, \tag{7.22}
$$

where $\gamma = \frac{\rho_1}{\rho}$ is the density ratio.

Proof. Using Equation (6.14), we calculate

$$c_0^2 = \frac{1}{2}(\omega^2)^{(2)} = g(1-\gamma)h_1 \tag{7.23}$$

as well as

$$\omega_1'(k_0) = \frac{\sqrt{g}}{2\sqrt{k_0}} = c_0 = \sqrt{g(1-\gamma)h_1} \implies k_0 = \frac{1}{4h_1(1-\gamma)}, \tag{7.24}$$

which completes the derivation. □

Using the resonance criteria from Equations (7.20) and (7.21), we simplify the modified Hamiltonian $\widehat{H}$.

Corollary 7.6. *The modified Hamiltonian* $\widehat{H} = H - c_0 I - \frac{\varepsilon_1^2}{\varepsilon}(\omega_1(k_0) - c_0 k_0)M$ *is*

$$\begin{aligned}
\widehat{H} &= H - c_0 I - \frac{\varepsilon_1^2}{\varepsilon}(\omega_1(k_0) - c_0 k_0)M \\
&= \int_{\mathbb{R}} \Bigg[\frac{\varepsilon}{2}\left(\widetilde{\mu} - c_0 \partial_X \widetilde{\zeta}\right)^2 + \varepsilon^2 \frac{(\omega^2)^{(3)}}{12}(\partial_X \widetilde{\zeta})(|D_X|\partial_X \widetilde{\zeta}) \\
&\qquad + \varepsilon^3 \frac{(\omega^2)^{(4)}}{48}\widetilde{\zeta}(D_X^4 \widetilde{\zeta}) + \frac{\varepsilon\varepsilon_1^2}{2}\omega_1''(k_0)\overline{v_1}(D_X^2 v_1) + \varepsilon^2 \kappa \widetilde{\mu}(D_X \widetilde{\zeta})^2 \\
&\quad + \varepsilon_1^2(\kappa_1 \widetilde{\mu} + \kappa_2 \partial_X \widetilde{\zeta})|v_1|^2 + \varepsilon^3 \kappa_3 \widetilde{\mu}(|D_X| D_X \widetilde{\zeta})(D_X \widetilde{\zeta}) \\
&\quad + \varepsilon\varepsilon_1^2(\kappa_4 \widetilde{\mu} + \kappa_5(\partial_X \widetilde{\zeta}))\left[v_1 \overline{D_X v_1} + \overline{v_1}(D_X v_1)\right] \\
&\quad + \varepsilon\varepsilon_1^2\big(\kappa_6(|D_X|\widetilde{\mu}) + \kappa_7(|D_X|\partial_X \widetilde{\zeta})\big)|v_1|^2 + \varepsilon^3 \kappa_8 (|D_X|\widetilde{\mu})(D_X \widetilde{\zeta})^2 \Bigg] dX \\
&\quad + \mathcal{O}(\varepsilon^4)
\end{aligned} \tag{7.25}$$

in the rescaled variables for the Benjamin-Ono and modulational regimes.

7.3 HAMILTONIAN IN CHARACTERISTIC VARIABLES

We examine the dynamics of the system in the principal direction of propagation. To do this, we adopt characteristic variables

$$\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2c_0}} & \sqrt{\frac{c_0}{2}} \\ \frac{1}{\sqrt{2c_0}} & -\sqrt{\frac{c_0}{2}} \end{pmatrix} \begin{pmatrix} \widetilde{\mu} \\ \widetilde{u} \end{pmatrix} =: \mathbf{O} \begin{pmatrix} \widetilde{\mu} \\ \widetilde{u} \end{pmatrix}, \tag{7.26}$$

where $\widetilde{u} = \partial_X \widetilde{\zeta}$, and $r(X,t)$ and $s(X,t)$ are the principally right-moving and left-moving parts of the solution, respectively. One may focus on the portion of the phase space in which $s = \mathcal{O}(\varepsilon^2)$ and where the wave propagates primarily to the right.

We effect the change of variables in Equation (7.26) to obtain the modified Hamiltonian $\widehat{H}$ in characteristic coordinates.

Proposition 7.7. The modified Hamiltonian $\widehat{H}$ is

$$\begin{aligned}\widehat{H} = \int_{\mathbb{R}} &\frac{(\omega^2)^{(3)}}{24c_0}\varepsilon^2 r(|D_X|r) - \frac{(\omega^2)^{(4)}}{96c_0}\varepsilon^3 r(\partial_X^2 r)\\ &- \frac{1}{2}\varepsilon\varepsilon_1^2\omega_1''(k_0)\overline{v_1}(\partial_X^2 v_1) - \kappa\varepsilon^2\frac{1}{2\sqrt{2c_0}}r^3\\ &+ \varepsilon_1^2\left(\kappa_1\sqrt{\frac{c_0}{2}} + \kappa_2\frac{1}{\sqrt{2c_0}}\right) r|v_1|^2 - (\kappa_3+\kappa_8)\frac{1}{2\sqrt{2c_0}}\varepsilon^3 r^2(|D_X|r)\\ &+ \varepsilon\varepsilon_1^2\left(\kappa_4\sqrt{\frac{c_0}{2}} + \kappa_5\frac{1}{\sqrt{2c_0}}\right) r\left[v_1(\overline{D_X v_1}) + \overline{v_1}(D_X v_1)\right]\\ &+ \varepsilon\varepsilon_1^2\left(\kappa_6\sqrt{\frac{c_0}{2}} + \kappa_7\frac{1}{\sqrt{2c_0}}\right)(|D_X|r)|v_1|^2\,dX + \mathcal{O}(\varepsilon^4)\end{aligned} \tag{7.27}$$

in characteristic variables $(r, s, v_1, \overline{v_1})$.

Proof. Since

$$s = \frac{1}{\sqrt{2c_0}}\widetilde{\mu} - \sqrt{\frac{c_0}{2}}\widetilde{u} \sim \mathcal{O}(\varepsilon^2), \tag{7.28}$$

it follows that

$$r = \frac{1}{\sqrt{2c_0}}\widetilde{\mu} + \sqrt{\frac{c_0}{2}}\widetilde{u} = \sqrt{\frac{2}{c_0}}\widetilde{\mu} + \mathcal{O}(\varepsilon^2). \tag{7.29}$$

This follows directly from the substitution of characteristic variables r and s into Corollary 7.6. □

Next we find the symplectic map J and Hamilton's equations of motion in the characteristic variables, r and s.

Proposition 7.8. Hamilton's equations and the symplectic map for characteristic variables (r, s) are given by

$$\frac{d}{dt}\begin{pmatrix} r \\ s \\ v_1 \\ \overline{v_1}\end{pmatrix} = \begin{pmatrix} -\partial_X & 0 & 0 & 0 \\ 0 & \partial_X & 0 & 0 \\ 0 & 0 & 0 & -i\varepsilon^{-1-2\delta} \\ 0 & 0 & i\varepsilon^{-1-2\delta} & 0\end{pmatrix}\begin{pmatrix} \delta_r\widehat{H} \\ \delta_s\widehat{H} \\ \delta_{v_1}\widehat{H} \\ \delta_{\overline{v_1}}\widehat{H}\end{pmatrix}, \tag{7.30}$$

where $\widehat{H}$ is the modified Hamiltonian given in Equation (7.27).

Proof. This follows readily from Lemma 7.3, since the change of variables matrix is orthogonal and so

$$\partial_t \begin{pmatrix} r \\ s \end{pmatrix} = \mathbf{O} \begin{pmatrix} 0 & -\partial_X \\ -\partial_X & 0 \end{pmatrix} \mathbf{O}^T \begin{pmatrix} \delta_r \widehat{H} \\ \delta_s \widehat{H} \end{pmatrix} = \begin{pmatrix} -\partial_X & 0 \\ 0 & \partial_X \end{pmatrix} \begin{pmatrix} \delta_r \widehat{H} \\ \delta_s \widehat{H} \end{pmatrix}, \tag{7.31}$$

which completes the proof. □

7.4 COUPLED SYSTEM OF EVOLUTION EQUATIONS

In this section, we find the coupled system of evolution equations of motion for the internal wave and the envelope of the free surface, using the symplectic map found in Equation (7.30) for long-time variable $\tau = \varepsilon^2 t$ for Benjamin-Ono as well as for Schrödinger in modulational regime $\varepsilon_1 = \varepsilon^{1+\delta}$.

Proposition 7.9. The free interface evolves according to a Benjamin-Ono equation

$$\begin{aligned}
\partial_\tau r = &-\frac{(\omega^2)^{(3)}}{12c_0}\partial_X(|D_X|r) + \varepsilon\frac{(\omega_1^2)^{(4)}}{48c_0}\partial_X^3 r + \frac{3\kappa}{\sqrt{2c_0}} r(\partial_X r) \\
&- \varepsilon^{2\delta}\left(\kappa_1\sqrt{\frac{c_0}{2}} + \kappa_2\frac{1}{\sqrt{2c_0}}\right)\partial_X(|v_1|^2) \\
&+ \varepsilon(\kappa_3+\kappa_8)\frac{1}{\sqrt{2c_0}}\left[\partial_X(r(|D_X|r)) + |D_X|(r\partial_X r)\right] \\
&- \varepsilon^{1+2\delta}\left(\kappa_4\sqrt{\frac{c_0}{2}} + \kappa_5\frac{1}{\sqrt{2c_0}}\right)\partial_X\left[v_1(\overline{D_X v_1}) + \overline{v_1}(D_X v_1)\right] \\
&- \varepsilon^{1+2\delta}\left(\kappa_6\sqrt{\frac{c_0}{2}} + \kappa_7\frac{1}{\sqrt{2c_0}}\right)\partial_X\big(|D_X|(|v_1|^2)\big),
\end{aligned} \tag{7.32}$$

coupled with the free surface, whose envelope satisfies a linear Schrödinger equation given by

$$\begin{aligned}
i\partial_\tau v_1 = &-\frac{\omega_1''(k_0)}{2}\partial_X^2 v_1 + \varepsilon^{-1}\left(\kappa_1\sqrt{\frac{c_0}{2}} + \kappa_2\frac{1}{\sqrt{2c_0}}\right) r v_1 \\
&- i\left(\kappa_4\sqrt{\frac{c_0}{2}} + \kappa_5\frac{1}{\sqrt{2c_0}}\right)(\partial_X(r v_1) + r\partial_X v_1) \\
&+ \left(\kappa_6\sqrt{\frac{c_0}{2}} + \kappa_7\frac{1}{\sqrt{2c_0}}\right) v_1(|D_X|r).
\end{aligned} \tag{7.33}$$

Proof. To derive the equations of motion for the interface and the free surface, we find the variational derivatives $\delta_r \widehat{H}$ and $\delta_{\overline{v_1}} \widehat{H}$ for the modified

Hamiltonian $\widehat{H}$, respectively, from Proposition 7.7 by acting on test function $v \in C_c^\infty(\mathbb{R})$. We show, for instance, that

$$\begin{aligned}
&\delta_r \left\{ \int_{\mathbb{R}} r^2(|D_X|r)\, dX \right\} [v] \\
&\quad = \lim_{\delta \to 0^+} \frac{1}{\delta} \int_{\mathbb{R}} (r+\delta v)^2(|D_X|(r+\delta v)) - r^2(|D_X|r)\, dX \\
&\quad = \int_{\mathbb{R}} \left[2rv(|D_X|r) + r^2(|D_X|v)\right] \, dX \\
&\quad = \left(\left(2r(|D_X|r) + |D_X|(r^2)\right), v\right)
\end{aligned} \tag{7.34}$$

and

$$\begin{aligned}
&\delta_{\overline{v_1}} \left\{ \int_{\mathbb{R}} r\left[v_1(\overline{D_X v_1}) + \overline{v_1}(D_X v_1)\right] dX \right\} [v] \\
&\quad = \int_{\mathbb{R}} r\left[v_1(\overline{D_X v}) + (D_X v_1)v\right] dX \\
&\quad = (D_X(rv_1) + r(D_X v_1), v)\,.
\end{aligned} \tag{7.35}$$

The variational derivatives are

$$\begin{aligned}
&\delta_r \widehat{H}[r] \\
&= \frac{(\omega^2)^{(3)}}{12c_0}(|D_X|r) - \frac{(\omega_1^2)^{(4)}(0)}{48c_0}\varepsilon^3(\partial_X^2 r) - \frac{3\kappa}{2\sqrt{2c_0}}\varepsilon^2 r^2 \\
&\quad + \varepsilon_1^2\left(\kappa_1\sqrt{\frac{c_0}{2}} + \kappa_2\frac{1}{\sqrt{2c_0}}\right)|v_1|^2 - \frac{\kappa_3+\kappa_8}{\sqrt{2c_0}}\varepsilon^3 r(|D_X|r) - \frac{\kappa_3+\kappa_8}{2\sqrt{2c_0}}\varepsilon^3|D_X|(r^2) \\
&\quad + \varepsilon\varepsilon_1^2\left(\kappa_4\sqrt{\frac{c_0}{2}} + \kappa_5\frac{1}{\sqrt{2c_0}}\right)\left[v_1(\overline{D_X v_1}) + \overline{v_1}(D_X v_1)\right] \\
&\quad + \varepsilon\varepsilon_1^2\left(\kappa_6\sqrt{\frac{c_0}{2}} + \kappa_7\frac{1}{\sqrt{2c_0}}\right)|D_X|(|v_1|^2)
\end{aligned} \tag{7.36}$$

and

$$\begin{aligned}
\delta_{\overline{v_1}}\widehat{H}[r] = &-\frac{1}{2}\varepsilon\varepsilon_1^2\omega_1''(k_0)\partial_X^2 v_1 + \varepsilon_1^2\left(\kappa_1\sqrt{\frac{c_0}{2}} + \kappa_2\frac{1}{\sqrt{2c_0}}\right)rv_1 \\
&- i\varepsilon\varepsilon_1^2\left(\kappa_4\sqrt{\frac{c_0}{2}} + \kappa_5\frac{1}{\sqrt{2c_0}}\right)(\partial_X(rv_1) + r\partial_X v_1) \\
&+ \varepsilon\varepsilon_1^2\left(\kappa_6\sqrt{\frac{c_0}{2}} + \kappa_7\frac{1}{\sqrt{2c_0}}\right)v_1(|D_X|r).
\end{aligned} \tag{7.37}$$

By Proposition 7.8, Hamilton's equation of motion are

$$\begin{aligned}
\partial_t r &= -\partial_X \delta_r \widehat{H} \\
&= -\partial_X \Bigg\{ \frac{(\omega^2)^{(3)}}{12c_0}\varepsilon^2(|D_X|r) - \frac{(\omega_1^2)^{(4)}}{48c_0}\varepsilon^3\partial_X^2 r - \frac{3\kappa}{2\sqrt{2c_0}}\varepsilon^2 r^2 \\
&\quad + \varepsilon_1^2\left(\kappa_1\sqrt{\frac{c_0}{2}} + \kappa_2\frac{1}{\sqrt{2c_0}}\right)|v_1|^2 - \frac{\kappa_3+\kappa_8}{\sqrt{2c_0}}\varepsilon^3 r(|D_X|r) - \frac{\kappa_3+\kappa_8}{2\sqrt{2c_0}}\varepsilon^3|D_X|(r^2) \\
&\quad + \varepsilon\varepsilon_1^2\left(\kappa_4\sqrt{\frac{c_0}{2}} + \kappa_5\frac{1}{\sqrt{2c_0}}\right)\left[v_1(\overline{D_X v_1}) + \overline{v_1}(D_X v_1)\right] \\
&\quad + \varepsilon\varepsilon_1^2\left(\kappa_6\sqrt{\frac{c_0}{2}} + \kappa_7\frac{1}{\sqrt{2c_0}}\right)|D_X|(|v_1|^2)\Bigg\},
\end{aligned} \tag{7.38}$$

and

$$\begin{aligned}
i\partial_t v_1 &= \varepsilon^{-1-2\delta}\delta_{\overline{v_1}} H \\
&= -\varepsilon^2\frac{\omega_1''(k_0)}{2}\partial_X^2 v_1 + \varepsilon\left(\kappa_1\sqrt{\frac{c_0}{2}} + \kappa_2\frac{1}{\sqrt{2c_0}}\right)rv_1 \\
&\quad - i\varepsilon^2\left(\kappa_4\sqrt{\frac{c_0}{2}} + \kappa_5\frac{1}{\sqrt{2c_0}}\right)(\partial_X(rv_1) + r\partial_X v_1) \\
&\quad + \varepsilon^2\left(\kappa_6\sqrt{\frac{c_0}{2}} + \kappa_7\frac{1}{\sqrt{2c_0}}\right)v_1(|D_X|r),
\end{aligned} \tag{7.39}$$

for the interface and free surface, respectively. After scaling, we yield the result. □

Observe that, truncated at cubic order, the Schrödinger equation for the envelope of the free surface is linear.

7.5 ANALYSIS OF COEFFICIENTS IN TERMS OF PHYSICAL PARAMETERS

Recalling that $c_0^2 = \frac{1}{2}(\omega^2)''(0) = g(1-\gamma)h_1$ and $k_0 = \frac{1}{4h_1(1-\gamma)}$ from Lemma 7.5, we find the explicit forms of the coefficients κ and κ_j for $j \in \{1,2,3,4,5,6\}$ from the proof of Proposition 6.6 in the previous chapter in terms of physical parameters g, h_1, ρ and ρ_1. This evaluation of the coefficients is needed to derive the higher-order Benjamin-Ono equation. Since some of the terms are $\lesssim \mathcal{O}(e^{-k_0 h_1})$, they make a negligible contribution to the equations of motion as $\gamma \to 1^-$ and $k_0 \to \infty$.

Proposition 7.10. The coefficients of the cubic terms of the Hamiltonian κ, κ_2, κ_3, κ_5, κ_7 and κ_8 from Equations (6.76), (6.78), (6.79), (6.81), (6.83) and (6.84) are

$$\begin{cases} \kappa &= \frac{\sqrt{g(1-\gamma)}}{2\sqrt{\rho_1}}, \\ \kappa_2 &= -\frac{\sqrt{g}}{4\sqrt{\rho_1(1-\gamma)}h_1} + \mathcal{O}(e^{-k_0h_1}) \\ \kappa_3 &= \frac{-\sqrt{g(1-\gamma)}\gamma h_1}{\sqrt{\rho_1}} \\ \kappa_5 &= -\frac{\sqrt{g(1-\gamma)}}{2\sqrt{\rho_1}} + \mathcal{O}(e^{-k_0h_1}) \\ \kappa_7 &= \frac{\sqrt{g}\gamma}{4\sqrt{\rho_1(1-\gamma)}} + \mathcal{O}(e^{-k_0h_1}) \\ \kappa_8 &= 0, \end{cases} \tag{7.40}$$

while the remaining ones from Equations (6.77), (6.80) and (6.82) scale like

$$\kappa_1, \kappa_4, \kappa_6 \lesssim \mathcal{O}(e^{-k_0h_1}). \tag{7.41}$$

Proof. To begin, we calculate the asymptotics of $\mathcal{B}_j$ for each $j \in \{1,2,3,4,5\}$ at resonant wavenumber $k = k_0$. First, we approximate

$$\begin{aligned} \mathcal{B}_1(k_0) &= \frac{b^-\sqrt{g(\rho-\rho_1)}G^{(0)}G_{11}^{(0)} + a^-\sqrt{g\rho_1}G^{(0)}G_{12}^{(0)}}{\rho G_{11}^{(0)} + \rho_1 G^{(0)}} \\ &\approx \frac{k_0}{\rho+\rho_1}\left(-\frac{\sqrt{\rho-\rho_1}}{\sqrt{\rho_1}}\sqrt{g(\rho-\rho_1)} - 2\sqrt{g\rho_1}\right)e^{-k_0h_1} \\ &= -\frac{k_0e^{-k_0h_1}\sqrt{g}}{\sqrt{\rho_1}}. \end{aligned} \tag{7.42}$$

Next, we approximate

$$\begin{aligned} \mathcal{B}_2(k_0) &= \frac{1}{\sqrt{g\rho_1}}\frac{a^-gG^{(0)}(\rho_1G_{11}^{(0)} + \rho G^{(0)}) + b^-g\sqrt{\rho_1(\rho-\rho_1)}G^{(0)}G_{12}^{(0)}}{\rho G_{11}^{(0)} + \rho_1 G^{(0)}} \\ &= \frac{1}{\sqrt{g\rho_1}}\left(gk_0 + \mathcal{O}(k_0e^{-2k_0h_1})\right) \approx \frac{\sqrt{g}k_0}{\sqrt{\rho_1}} \end{aligned} \tag{7.43}$$

Then, we find $\mathcal{B}_3 = \mathcal{B}_1\mathrm{sgn}(k_0)$,

$$\mathcal{B}_4(k_0) = b^-\sqrt{g(\rho-\rho_1)}DG^{(0)}B_0^{-1} - \frac{\rho}{\rho_1}a^-\sqrt{g\rho_1}DB_0^{-1}G_{12}^{(0)} \lesssim \mathcal{O}(k_0e^{-k_0h_1}) \tag{7.44}$$

and finally $\mathcal{B}_5(k_0) = -a^- \sqrt{g\rho_1}k_0 \approx -k_0\sqrt{g\rho_1}$. The second step is to do the formal calculations. We recall the coefficients of the grouped terms deduced from R_j for each $j \in \{1,2,3,4,5\}$. We recall that

$$\kappa = \frac{\sqrt{g(1-\gamma)}}{2\sqrt{\rho_1}}, \quad \kappa_3 = \frac{-\sqrt{g(1-\gamma)}\gamma h_1}{\sqrt{\rho_1}}, \quad \kappa_8 = 0 \tag{7.45}$$

from our calculations at the beginning of this section. Next, noting that $b^+(0) = \sqrt{\gamma}$ and $a^+(0) = \sqrt{1-\gamma}$, we calculate

$$\begin{aligned}\kappa_1 &= \frac{1}{2}\sqrt{\frac{\rho_1}{g}}a^+(0)(\mathcal{B}_2^2\omega_1^{-1})(k_0) - \frac{1}{2\rho_1\sqrt{g\rho_1}}a^+(0)(\mathcal{B}_5^2\omega_1^{-1})(k_0) + \mathcal{O}(e^{-k_0h_1}) \\ &= \frac{k_0^{3/2}\sqrt{1-\gamma}}{2\sqrt{\rho_1}} - \frac{k_0^{3/2}\sqrt{1-\gamma}}{2\sqrt{\rho_1}} + \mathcal{O}(e^{-k_0h_1}) \lesssim \mathcal{O}(e^{-k_0h_1}),\end{aligned} \tag{7.46}$$

$$\begin{aligned}\kappa_2 &= \frac{\sqrt{1-\gamma}}{\rho_1}(a^-\mathcal{B}_5)(k_0) + \mathcal{O}(e^{-k_0h_1}) \\ &= \frac{-k_0\sqrt{g(1-\gamma)}}{\sqrt{\rho_1}} + \mathcal{O}(e^{-k_0h_1}) = -\frac{\sqrt{g}}{4\sqrt{\rho_1(1-\gamma)}h_1} + \mathcal{O}(e^{-k_0h_1}),\end{aligned} \tag{7.47}$$

$$\begin{aligned}\kappa_4 &= \frac{\sqrt{\rho_1}a^+(0)}{4\sqrt{g}}(\mathcal{B}_2^2\omega_1^{-1})'(k_0) - \frac{a^+(0)}{4\rho_1\sqrt{g\rho_1}}(\mathcal{B}_5^2\omega_1^{-1})'(k_0) + \mathcal{O}(e^{-k_0h_1}) \\ &= \frac{\sqrt{\rho_1(1-\gamma)}}{4\sqrt{g}}\partial_k\left(\frac{gk^{3/2}}{\rho_1\sqrt{g}}\right)(k_0) \\ &\quad - \frac{1}{4\rho_1\sqrt{g\rho_1}}\frac{\sqrt{\rho-\rho_1}}{\sqrt{\rho}}\partial_k\left(\frac{k^{3/2}g\rho_1}{\sqrt{g}}\right)(k_0) + \mathcal{O}(e^{-k_0h_1}) \\ &\lesssim \mathcal{O}(e^{-k_0h_1})\end{aligned} \tag{7.48}$$

and

$$\begin{aligned}\kappa_5 &= -\frac{1}{2\rho_1\sqrt{g\rho_1}}\mathcal{A}_5^{(1)}(0)(a^-\mathcal{B}_5)'(k_0) + \mathcal{O}(e^{-k_0h_1}) \\ &= -\frac{1}{2\rho_1\sqrt{g\rho_1}}(-\sqrt{g\rho_1}a^+(0))\partial_k(-k\sqrt{g\rho_1})(k_0) + \mathcal{O}(e^{-k_0h_1}) \\ &= -\frac{\sqrt{g(1-\gamma)}}{2\sqrt{\rho_1}} + \mathcal{O}(e^{-k_0h_1}).\end{aligned} \tag{7.49}$$

Similarly to the computation for κ_1, we show

$$\kappa_6 = \frac{1}{2}\frac{\sqrt{\rho_1}}{\sqrt{g}}(a^+)^{(1)}(\mathcal{B}_2^2\omega_1^{-1})(k_0) - \frac{1}{2\rho_1\sqrt{g\rho_1}}(\mathcal{B}_5^2\omega_1^{-1})(k_0)(a^+)^{(1)} + \mathcal{O}(e^{-k_0h_1}) \lesssim \mathcal{O}(e^{-k_0h_1}). \tag{7.50}$$

Lastly, we verify

$$\begin{aligned}\kappa_7 &= -\frac{1}{\rho_1\sqrt{g\rho_1}}(a^-\mathcal{B}_5)(k_0)\mathcal{A}_5^{(1,1)} + \mathcal{O}(e^{-k_0h_1}) \\ &= \frac{1}{\rho_1\sqrt{g\rho_1}}(\sqrt{g\rho_1}k_0)\big(-\sqrt{g\rho_1}(a^+)^{(1)}\big) + \mathcal{O}(e^{-k_0h_1}) \\ &= \frac{\sqrt{g}\gamma}{4\sqrt{\rho_1(1-\gamma)}} + \mathcal{O}(e^{-k_0h_1}),\end{aligned} \tag{7.51}$$

as both $\mathcal{B}_3(k_0) \lesssim \mathcal{O}(e^{-k_0h_1})$ and $\mathcal{B}_4(k_0) \lesssim \mathcal{O}(k_0e^{-k_0h_1})$, which completes the calculation. □

Lastly, we write the coupled system of evolution equations in terms of the relevant coefficients.

Proposition 7.11. Neglecting the exponential terms of order $\mathcal{O}(e^{-k_0h_1})$ from κ_1, κ_4 and κ_6, the coupled system has the form of a higher-order Benjamin-Ono equation for the free interface

$$\begin{aligned}\partial_\tau r = &-\frac{(\omega^2)^{(3)}}{12c_0}\partial_X(|D_X|r) + \varepsilon\frac{(\omega^2)^{(4)}(0)}{48c_0}\partial_X^3 r + \frac{3\kappa}{\sqrt{2c_0}}r(\partial_X r) \\ &- \varepsilon^{2\delta}\frac{\kappa_2}{\sqrt{2c_0}}\partial_X(|v_1|^2) + \varepsilon\frac{\kappa_3}{\sqrt{2c_0}}\left[\partial_X(r(|D_X|r)) + |D_X|(r\partial_X r)\right] \\ &- \varepsilon^{1+2\delta}\frac{\kappa_5}{\sqrt{2c_0}}\partial_X\left[v_1(\overline{D_X v_1}) + \overline{v_1}(D_X v_1)\right] \\ &- \varepsilon^{1+2\delta}\frac{\kappa_7}{\sqrt{2c_0}}\partial_X\big(|D_X|(|v_1|^2)\big),\end{aligned} \tag{7.52}$$

and a linear Schrödinger equation for the free surface

$$\begin{aligned}i\partial_\tau v_1 = &-\frac{\omega_1''(k_0)}{2}\partial_X^2 v_1 + \varepsilon^{-1}\frac{\kappa_2}{\sqrt{2c_0}}rv_1 - i\frac{\kappa_5}{\sqrt{2c_0}}(\partial_X(rv_1) + r\partial_X v_1) \\ &+ \frac{\kappa_7}{\sqrt{2c_0}}v_1(|D_X|r),\end{aligned} \tag{7.53}$$

where the coefficients are expressed in terms of the physical parameters g, ρ, ρ_1 and h_1.

Proof. This follows directly from Propositions 7.9 and 7.10. □

7.6 CONCLUSION AND FUTURE WORK

We performed asymptotic analysis of the coupling of free internal and surface modes for a fluid consisting of two layers, the lower of which is infinitely deep. We have treated the internal mode under the Benjamin-Ono scaling regime, while the surface is approximated by a modulated, quasi-monochromatic wave. This modulation appropriately corresponds to the "ripple effect" that occurs in natural ocean dynamics [PS65], in which the visible tides on the surface and the internal wave propagate at the same velocity. Using a Hamiltonian formulation of the water wave problem and perturbation theory, as developed by [CS93], [CGK05] and [CGS11], we derived a system of evolution equations: the free interface evolves according to a higher-order Benjamin-Ono equation and is coupled to the free surface, whose envelope satisfies a time-dependent, linear Schrödinger equation.

When high-order corrections are neglected, Guo-Miao [GM99] and Pecher [Pec06] proved global well-posedness of KdV and BO equations, respectively, coupled to a linear Schrödinger equation. Specifically, Pecher proved that dispersive system

$$\begin{cases} \partial_\tau r &= \alpha_1 \partial_X(|D_X| r) - \varepsilon^{2\delta}\alpha_2 \partial_X(|v_1|^2) \\ i\partial_\tau v_1 &= -\beta_1 \partial_X^2 v_1 + \varepsilon^{-1}\beta_2 r v_1 \end{cases} \tag{7.54}$$

with initial data $r(x,0) = r_0(x)$ and $v_1(x,0) = (v_1)_0(x)$ is globally well-posed. More recently, Linares, Pilod and Ponce [LPP11] established local well-posedness for a higher-order Benjamin-Ono equation

$$\begin{cases} \partial_t u - b\mathcal{H}(u_{xx}) + a u_{xxx} &= c u u_x - d\left[u\mathcal{H}(u_x) + \mathcal{H}(u u_x)\right]_x \\ u(x,0) = u_0(x) \end{cases} \tag{7.55}$$

with initial data $u_0 \in H^s$, $s \geqslant 2$. Currently, we are working to extend this result to a higher-order coupled system

$$\begin{cases} \partial_\tau r &= \alpha_1 \partial_X(|D_X| r) + \varepsilon\alpha_2 \partial_X^3 r + \alpha_3 r(\partial_X r) \\ &\quad -\varepsilon^{2\delta}\alpha_4 \partial_X(|v_1|^2) + \varepsilon\alpha_5\left[\partial_X(r(|D_X| r)) + |D_X|(r\partial_X r)\right] \\ i\partial_\tau v_1 &= -\beta_1 \partial_X^2 v_1 + \varepsilon^{-1}\beta_2 r v_1. \end{cases} \tag{7.56}$$

I am also considering the extension of the analysis from two dimensions to the three-dimensional water wave problem. Coupling between internal and gravity surface waves has previously been studied for three-dimensional fluids comprised of two distinct layers [CGS15] in the shallow water limit, in which the equation of motion for the interface is the Kadomtsev–Petviashvili II (KP II) equation [KP70]. In the deep water model, one of the earlier higher dimensional versions of the BO equation is Shrira's equation [Shr89]

$$\partial_t u - \mathcal{R}_1 u_{xx} + \frac{1}{2}(u^2)_x = 0\,, \qquad (x,y) \in \mathbb{R}^2, t \in \mathbb{R}, \tag{7.57}$$

where the operator $\mathcal{R}_1$ denotes the Riesz transform with respect to the first variable defined by

$$(\mathcal{R}_1 f)(x,y) := \frac{1}{2\pi}\text{p.v.}\left\{\iint\limits_{\mathbb{R}^2} \frac{(x-z_1)f(z_1,z_2)}{((x-z_1)^2+(y-z_2)^2)^{3/2}}\,dz_1\,dz_2\right\}. \tag{7.58}$$

We expect that the long internal wave will evolve according to a two-dimensional Benjamin-Ono (2D BO) equation

$$\begin{cases} \partial_t u - \mathcal{H}(u_{xx}+u_{yy}) + \frac{1}{2}(u^2)_x = 0 \\ u(x,y,0) = u_0(x,y), \end{cases} \tag{7.59}$$

which was derived precisely as the KP II approximation to the BO equation [Nas23]. Interestingly, both the KP II and 2D BO equations have similar forms as studied by Ablowitz, Demirci and Ma [ADM16], which is an avenue for analysis of the model equations in deep water to explain the characteristic features physically observed on the surface of seas due to the internal modes.

Part II

A BOCHNER FORMULA ON PARABOLIC PATH SPACE FOR THE RICCI FLOW

1 INTRODUCTION

The goal of this part of the thesis is to prove a Bochner formula on path space for the Ricci flow, and to discuss some applications. This generalizes the Bochner formula on path space for Einstein metrics from Haslhofer and Naber [HN18b].

Throughout this part, we shall use the convention that an evolving family of manifolds is a smooth and complete family of Riemannian manifolds $(M^n, g_t)_{t\in I}$ such that

$$\sup_{M\times I} \left(|\mathrm{Rm}| + |\partial_t g| + |\nabla \partial_t g|\right) < \infty. \tag{1.1}$$

1.1 BACKGROUND ON CHARACTERIZATIONS OF EINSTEIN METRICS

To begin, let us recall some well-known characterizations of when a Riemannian manifold (M, g) is a supersolution to the Einstein equations. Let $H_t f$ denote the heat flow of a function $f : M \to \mathbb{R}$. Then its gradient satisfies the Bochner formula

$$(-\partial_t + \Delta)|\nabla H_t f|^2 = 2|\nabla^2 H_t f|^2 + 2\mathrm{Rc}(\nabla H_t f, \nabla H_t f). \tag{1.2}$$

Using this, an equivalence between supersolutions of the Einstein equations, the classical Bochner inequality and the gradient estimate readily follows, i.e.

$$\mathrm{Rc} \geqslant 0 \iff (\partial_t - \Delta)|\nabla H_t f|^2 \leqslant -2|\nabla^2 H_t f|^2 \tag{1.3}$$

$$\iff |\nabla H_t f| \leqslant H_t |\nabla f|, \tag{1.4}$$

for all test functions $f : M \to \mathbb{R}$.

Until recently, however, there was no analogous characterization of solutions to the Einstein equations. Such a characterization was discovered by Naber [Nab13] by employing the analytic properties of path space $PM = C([0, \infty), M)$. This path space is naturally endowed with a family of Wiener measures $\{\mathbb{P}_x\}$ of Brownian motion starting at $x \in M$. One then introduces a notion of stochastic parallel transport and the corresponding family of parallel gradients $\{\nabla_s^{||}\}$. Using this foundation, Naber [Nab13] developed an infinite-dimensional generalization of the gradient estimate (1.4) to characterize solutions of the Einstein equations. Namely, he proved that

$$\mathrm{Rc} = 0 \iff \left|\nabla_x \int_{PM} F \, d\mathbb{P}_x\right| \leqslant \int_{PM} |\nabla_0^{||} F| \, d\mathbb{P}_x, \tag{1.5}$$

for all test functions $F : PM \to \mathbb{R}$.

Interesting variants of these characterizations and estimates have been obtained in [CT18b], [CT18a], [Wu20], [FW17] and [WW18].

Later, Haslhofer and Naber [HN18b] proved an infinite-dimensional generalization of (1.3). Namely, they showed

$$\mathrm{Rc} = 0 \iff d|\nabla_s F_t|^2 \geqslant \langle \nabla_t |\nabla_s F_t|^2, dW_t \rangle \tag{1.6}$$

for all martingales $F_t : PM \to \mathbb{R}$.

Using the infinite-dimensional Bochner formula (1.6), they gave a simpler proof of the infinite-dimensional gradient estimate (1.5) in a similar vein to how the classical Bochner formula (1.3) readily implies the classical gradient estimate (1.4).

1.2 BACKGROUND ON CHARACTERIZATIONS OF RICCI FLOW

To motivate the characterization of solutions of the Ricci flow, let us first recall characterizations of supersolutions, namely evolving Riemannian manifolds $(M, g_t)_{t \in I}$ such that

$$\partial_t g_t \geqslant -2\mathrm{Rc}_{g_t}. \tag{1.7}$$

To begin, consider the heat flow $H_{st} f$ on this evolving background, namely the solution of the heat equation $\partial_t u = \Delta_{g_t} u$ with initial condition f at time $t = s$. Then its gradient satisfies the Bochner formula

$$(\partial_t - \Delta_{g_t}) |\nabla H_{st} f|^2 = -2|\nabla^2 H_{st} f|^2 + (\partial_t g_t + 2\mathrm{Rc}_{g_t})(\nabla H_{st} f, \nabla H_{st} f). \tag{1.8}$$

Using this, an equivalence between supersolutions of the Ricci flow, the Bochner inequality and the gradient estimate readily follows, i.e.

$$\partial_t g_t \geqslant -2\mathrm{Rc}_{g_t} \iff (\partial_t - \Delta_{g_t})|\nabla H_{st} f|^2 \leqslant -2|\nabla^2 H_{st} f|^2 \tag{1.9}$$
$$\iff |\nabla H_{st} f| \leqslant H_{st}|\nabla f|, \tag{1.10}$$

for all test functions $f : M \to \mathbb{R}$.

To generalize the inequality (1.10) to an infinite dimensional estimate, Haslhofer and Naber [HN18a] considered space-time $\mathcal{M} = M \times I$ equipped with the space-time connection defined on vector fields by

$$\nabla_X Y = \nabla_X^{g_t} Y, \qquad \nabla_t Y = \partial_t Y + \frac{1}{2}\partial_t g_t(Y, \cdot)^{\#g_t} \tag{1.11}$$

The main difference, compared to the infinite dimensional estimate that characterizes Einstein metrics, is that the parabolic path space $P_T\mathcal{M}$ only consists of continuous space-time curves $\{\gamma_\tau = (T - \tau, x_\tau)\}$ that move backwards along the time-axis with unit speed and start at fixed time $T \in I$. This path space is naturally endowed with a family of parabolic Wiener measures $\{\mathbb{P}_{(x,T)}\}$ of Brownian motion starting at $(x, T) \in \mathcal{M}$ and parabolic stochastic parallel gradients $\{\nabla_\sigma^{||}\}_{\sigma \geqslant 0}$ defined via (1.11). Using this framework, Haslhofer and Naber proved an infinite-dimensional generalization of the gradient estimate (1.10) that characterizes solutions of the Ricci flow. Namely, they proved that

$$\partial_t g_t = -2\mathrm{Rc}_{g_t} \iff \left|\nabla_x \int_{P_T\mathcal{M}} F \, d\mathbb{P}_{(x,T)}\right| \leqslant \int_{P_T\mathcal{M}} |\nabla_0^{||} F| \, d\mathbb{P}_{(x,T)} \tag{1.12}$$

for all test functions $F : P_T\mathcal{M} \to \mathbb{R}$.

Some nice variants of these characterizations have been obtained by Cheng and Thalmaier [CT18a]. Moreover, Cabezas-Rivas and Haslhofer [CRH20] found an interesting link between estimates in the elliptic and parabolic settings.

However, there is no analogous treatment of the Bochner inequality (1.6) in the time-dependent setting. The primary goal of this part shall be to prove such an equivalent notion.

1.3 BOCHNER FORMULA ON PARABOLIC PATH SPACE

Let $(M, g_t)_{t\in I}$ be a family of evolving manifolds and let $\mathcal{M} = M \times I$ be its space-time equipped with the space-time connection defined on vector fields via (1.11). Next, as in Section 1.2, we consider the parabolic path space $P_T\mathcal{M}$, given by

$$P_T\mathcal{M} := \left\{ (x_\tau, T-\tau)_{\tau\in[0,T]} | x \in C([0,T], M) \right\}, \tag{1.13}$$

and endow this space with the parabolic Wiener measure of Brownian motion on space-time, $\mathbb{P}_{(x,T)}$, based at $(x,T) \in \mathcal{M}$ as well as the associated parabolic parallel gradients $\nabla_\sigma^{||}$ defined via stochastic parallel transport on space-time $\mathcal{M}$. To explain these notions in more detail, first recall that the solution to the heat equation $\partial_t u = \Delta_{g_t} u$ with initial condition f at time $t = s$ is given by convolving with the heat kernel i.e.

$$H_{st} f(x) = \int_M H(x,t|y,s) f(y)\, dV_{g_s}(y). \tag{1.14}$$

The Wiener measure $\mathbb{P}_{(x,T)}$ is then uniquely characterized in terms of the heat kernel by

$$\begin{aligned} &\mathbb{P}_{(x,T)} \left[X_{\tau_1} \in U_1, \ldots, X_{\tau_k} \in U_k \right] \\ &= \int_{U_1} \cdots \int_{U_k} H(x,T|x_1,T-\tau_1) \cdots H(x_{k-1},T-\tau_{k-1}|x_k,T-\tau_k) d\mathrm{Vol}_{g_{T-\tau_1}}(x_1) \cdots d\mathrm{Vol}_{g_{T-\tau_k}}(x_k) \end{aligned} \tag{1.15}$$

where X_τ is a Brownian motion on M starting at x. Moreover, the stochastic parallel gradient $\nabla_\sigma^{||} F(\gamma) \in (T_x M, g_T)$ of a function $F : P_T\mathcal{M} \to \mathbb{R}$, is expressed in terms of the Fréchet derivative by

$$D_{V^\sigma} F(\gamma) = \langle \nabla_\sigma^{||} F(\gamma), v \rangle_{(T_x M, g_T)}, \tag{1.16}$$

where V^σ is the vector field along γ defined by $V^\sigma_\tau = P_\tau^{-1} v \mathbb{1}_{[\sigma,T]}(\tau)$ and $\{P_\tau\}$, a family of isometries, referred to as stochastic parallel transport.

With the aim of generalizing (1.9) to an infinite-dimensional estimate, we consider martingales on parabolic path space, i.e. Σ_τ-adapted integrable processes $F_\tau : P_{(x,T)}\mathcal{M} \to \mathbb{R}$ that satisfy

$$F_{\tau_1} = \mathbb{E}[F_{\tau_2} | \Sigma_{\tau_1}], \tag{1.17}$$

where $\mathbb{E}[\cdot \,|\, \Sigma_\tau]$ denotes the conditional expectation with respect to the σ-algebra Σ_τ of events observable at time τ.

For example, if $F(\gamma) = f(\pi_1\gamma_{\tau_1})$, where $f : M \to \mathbb{R}$ and $\pi_1 : M \times I \to I$, then the induced martingale $F_\tau = \mathbb{E}[F \,|\, \Sigma_\tau]$ for $\tau < \tau_1$ is given by

$$F_\tau(\gamma) = H_{T-\tau_1, T-\tau} f(\pi_1\gamma_\tau) \qquad \text{(see example 2.18)}. \tag{1.18}$$

Specifically, martingales generalize heat flow. This analogue between martingales and heat kernels will motivate our development of the following generalized Bochner formula on $P\mathcal{M}$.

Theorem 1.1. *(Generalized Bochner Formula on $P\mathcal{M}$) Let $F_\tau : P_{(x,T)}\mathcal{M} \to \mathbb{R}$ be a martingale on the parabolic path space of space-time. If $\sigma \geqslant 0$ is fixed, then*

$$\begin{aligned} d(|\nabla^{||}_\sigma F_\tau|^2) = \langle \nabla^{||}_\tau |\nabla^{||}_\sigma F_\tau|^2, dW_\tau\rangle + (\dot{g} + 2\mathrm{Rc})_\tau(\nabla^{||}_\tau F_\tau, \nabla^{||}_\sigma F_\tau)\, d\tau \\ + 2|\nabla^{||}_\tau \nabla^{||}_\sigma F_\tau|^2\, d\tau + 2|\nabla^{||}_\sigma F_\sigma|^2\, d\delta_\sigma(\tau), \end{aligned} \tag{1.19}$$

where $(\dot{g} + 2\mathrm{Rc})_\tau(v, w) = (\dot{g}_t + 2\mathrm{Rc}_{g_t})|_{t=T-\tau}(P_\tau^{-1}v, P_\tau^{-1}w)$ and $\dot{g} = \frac{d}{dt}g$.

This generalized Bochner formula proves to be a fundamental tool in characterizing the Ricci flow. Note that, if $(M, g_t)_{t\in I}$ evolves by Ricci flow, this formula reduces to

$$d(|\nabla^{||}_\sigma F_\tau|^2) = \langle \nabla^{||}_\tau |\nabla^{||}_\sigma F_\tau|^2, dW_\tau\rangle + 2|\nabla^{||}_\tau \nabla^{||}_\sigma F_\tau|^2\, d\tau + 2|\nabla^{||}_\sigma F_\sigma|^2\, d\delta_\sigma(\tau), \tag{1.20}$$

and this trivially implies the following infinite-dimensional generalization of Bochner inequality (1.9) in the time-dependent setting

$$d(|\nabla^{||}_\sigma F_\tau|^2) \geqslant \langle \nabla^{||}_\tau |\nabla^{||}_\sigma F_\tau|^2, dW_\tau\rangle + 2|\nabla^{||}_\tau \nabla^{||}_\sigma F_\tau|^2\, d\tau + 2|\nabla^{||}_\sigma F_\sigma|^2\, d\delta_\sigma(\tau). \tag{1.21}$$

In contrast to the heat flow Bochner inequality, this generalized martingale Bochner inequality (1.21) as well as the estimates that follow from it are strong enough to help exhibit solutions and not just supersolutions of the Ricci flow.

Specifically, Theorem 1.1 has four main applications:

- a characterization of the Ricci flow via Bochner inequalities for martingales on parabolic path space;
- gradient estimates for martingales on parabolic path space;
- Hessian estimates for martingales on parabolic path space;

- a new and much simpler proof of the characterization of solutions of the Ricci flow by Haslhofer and Naber in 2018 [HN18a, Theorem 1.22],

which will be discussed in Section 1.4.

To explain the meaning of Theorem 1.1 in the simplest example, this generalized Bochner formula on $P\mathcal{M}$ directly reduces to the standard Bochner formula in the case of 1-point functions, i.e. when $F_\tau(\gamma)$ satisfies equation (1.18). That is, the evolution of $|\nabla H_{T-\tau_1,T-\tau} f|^2$ for $\tau \leqslant \tau_1$ is calculated as

$$\left(-\partial_\tau - \Delta_{g_{T-\tau}}\right) |\nabla H_{T-\tau_1,T-\tau} f|^2 \leqslant -2|\nabla^2 H_{T-\tau_1,T-\tau} f|^2 \tag{1.22}$$

in Corollary 3.4. Setting $s = T - \tau_1$ and $t = T - \tau$, this explicitly recovers (1.9) from Section 1.2.

1.4 APPLICATIONS

We will conclude with some main applications of our Bochner inequality (1.21). First, we shall develop a new characterization of the Ricci flow.

Theorem 1.2. *(New characterizations of the Ricci Flow) For an evolving family of manifolds $(M^n, g_t)_{t\in I}$, the following are equivalent to solving the Ricci flow $\partial_t g_t = -2\mathrm{Rc}_{g_t}$:*

(C1) *Martingales on parabolic path space satisfy the full Bochner inequality*

$$d|\nabla^{||}_\sigma F_\tau|^2 \geqslant \langle \nabla_\tau |\nabla^{||}_\sigma F_\tau|^2, dW_\tau \rangle + 2|\nabla^{||}_\tau \nabla^{||}_\sigma F_\tau|^2\, d\tau + 2|\nabla^{||}_\sigma F_\sigma|^2\, d\delta_\sigma(\tau) \tag{1.23}$$

(C2) *Martingales on parabolic path space satisfy the dimensional Bochner inequality*

$$d|\nabla^{||}_\sigma F_\tau|^2 \geqslant \langle \nabla_\tau |\nabla^{||}_\sigma F_\tau|^2, dW_\tau \rangle + \frac{2}{n}|\Delta^{||}_{\sigma,\tau} F_\tau|^2\, d\tau + 2|\nabla^{||}_\sigma F_\sigma|^2\, d\delta_\sigma(\tau) \tag{1.24}$$

(C3) *Martingales on parabolic path space satisfy the weak Bochner inequality*

$$d|\nabla^{||}_\sigma F_\tau|^2 \geqslant \langle \nabla_\tau |\nabla^{||}_\sigma F_\tau|^2, dW_\tau \rangle + 2|\nabla^{||}_\sigma F_\sigma|^2\, d\delta_\sigma(\tau) \tag{1.25}$$

(C4) *Martingales on parabolic path space satisfy the linear Bochner inequality*

$$d|\nabla^{||}_\sigma F_\tau| \geqslant \langle \nabla_\tau |\nabla^{||}_\sigma F_\tau|, dW_\tau \rangle + |\nabla^{||}_\sigma F_\sigma|\, d\delta_\sigma(\tau) \tag{1.26}$$

(C5) *If F_τ is a martingale, then $\tau \to |\nabla^{||}_\sigma F_\tau|$ is a submartingale for every $\sigma \geqslant 0$.*

Second, we shall obtain gradient estimates for martingales on parabolic path space.

Theorem 1.3. *(Gradient Estimates for Martingales on Parabolic Path Space) For an evolving family of manifolds $(M^n, g_t)_{t\in I}$, the following are equivalent to solving the Ricci flow $\partial_t g_t = -2\mathrm{Rc}_{g_t}$:*

(G1) *For any $F \in L^2(P\mathcal{M})$, σ fixed and $\tau_1 \leqslant \tau_2$, the induced martingale satisfies the gradient estimate*

$$|\nabla^{||}_\sigma F_{\tau_1}| \leqslant \mathbb{E}_{(x,T)}\left[|\nabla^{||}_\sigma F_{\tau_2}||\Sigma_{\tau_1}\right]. \tag{1.27}$$

(G2) *For any $F \in L^2(P\mathcal{M})$, σ fixed and $\tau_1 \leqslant \tau_2$, the induced martingale satisfies the gradient estimate*

$$|\nabla^{||}_\sigma F_{\tau_1}|^2 \leqslant \mathbb{E}_{(x,T)}\left[|\nabla^{||}_\sigma F_{\tau_2}|^2|\Sigma_{\tau_1}\right]. \tag{1.28}$$

Note that in the case of $\sigma = \tau_1 = 0$, (G1) reduces to the infinite-dimensional gradient estimate (1.12).

Next, we shall obtain Hessian estimates for martingales on parabolic path space.

Theorem 1.4. *(Hessian Estimates for Martingales on Parabolic Path Space) For an evolving family of manifolds $(M^n, g_t)_{t\in I}$ that solve the Ricci flow $\partial_t g_t = -2\mathrm{Rc}_{g_t}$ and a function $F \in L^2(P\mathcal{M})$, it holds that:*

(H1) *For each $\sigma \geqslant 0$, we have the estimate*

$$\mathbb{E}_{(x,T)}\left[|\nabla^{||}_\sigma F_\sigma|^2\right] + 2\mathbb{E}_{(x,T)}\int_0^T \left[|\nabla^{||}_\tau \nabla^{||}_\sigma F_\tau|^2\right] d\tau \leqslant \mathbb{E}_{(x,T)}\left[|\nabla^{||}_\sigma F|^2\right]. \tag{1.29}$$

(H2) *We have the Poincaré Hessian estimate*

$$\mathbb{E}_{(x,T)}\left[\left(F - \mathbb{E}_{(x,T)}[F]\right)^2\right] + 2\int_0^T\int_0^T \mathbb{E}_{(x,T)}\left[|\nabla^{||}_\tau \nabla^{||}_\sigma F_\tau|^2\right] d\sigma\, d\tau \leqslant \int_0^T \mathbb{E}_{(x,T)}\left[|\nabla^{||}_\sigma F|^2\right] d\sigma. \tag{1.30}$$

(H3) *We have the log-Sobolev Hessian estimate*

$$\begin{aligned}&\mathbb{E}_{(x,T)}\left[F^2\ln(F^2)\right] - \mathbb{E}_{(x,T)}[F^2]\ln\left(\mathbb{E}_{(x,T)}[F^2]\right)\\ &\quad + 2\int_0^T\int_0^T \mathbb{E}_{(x,T)}\left[(F^2)_\tau|\nabla^{||}_\tau\nabla^{||}_\sigma \ln((F^2)_\tau)|^2\right]d\sigma\, d\tau\\ &\quad \leqslant 4\int_0^T \mathbb{E}_{(x,T)}\left[|\nabla^{||}_\sigma F|^2\right]d\sigma. \end{aligned} \tag{1.31}$$

Finally, our generalized Bochner formula on parabolic path space leads to a simpler proof of the characterization of solutions of the Ricci flow found by Haslhofer and Naber [HN18a].

Theorem 1.5. *(Characterization of Solutions of the Ricci Flow) [HN18a, Theorem 1.22] For an evolving family of manifolds $(M^n, g_t)_{t\in I}$, the following are equivalent:*

(R1) *$(M^n, g_t)_{t\in I}$ solves the Ricci flow $\partial_t g_t = -2\mathrm{Rc}_{g_t}$.*

(R2) *For every $F \in L^2(P\mathcal{M})$, we have the gradient estimate*

$$\left|\nabla_x \mathbb{E}_{(x,T)}[F]\right| \leqslant \mathbb{E}_{(x,T)}[|\nabla^{||}_0 F|]. \tag{1.32}$$

(R3) *For every $F \in L^2(P\mathcal{M})$, the induced martingale $\{F_\tau\}_{\tau\in[0,T]}$ satisfies the quadratic variation estimate*

$$\mathbb{E}_{(x,T)}\left[\frac{d[F,F]_\tau}{d\tau}\right] \leqslant 2\mathbb{E}_{(x,T)}\left[|\nabla^{||}_\tau F|^2\right]. \tag{1.33}$$

(R4) *The Ornstein-Uhlenbeck operator $\mathcal{L}_{(\tau_1,\tau_2)}$ on parabolic path space $L^2(P\mathcal{M})$ satisfies the log-Sobolev inequality*

$$\begin{aligned}&\mathbb{E}_{(x,T)}\left[(F^2)_{\tau_2}\log((F^2)_{\tau_2}) - (F^2)_{\tau_1}\log((F^2)_{\tau_1})\right]\\ &\quad\leqslant 2\mathbb{E}_{(x,T)}\left[\langle F, \mathcal{L}_{(\tau_1,\tau_2)}F\rangle_{\mathcal{H}}\right]. \end{aligned} \tag{1.34}$$

(R5) *The Ornstein-Uhlenbeck operator $\mathcal{L}_{(\tau_1,\tau_2)}$ on parabolic path space $L^2(P\mathcal{M})$ satisfies the Poincaré inequality*

$$\mathbb{E}_{(x,T)}\left[(F_{\tau_2} - F_{\tau_1})^2\right] \leqslant \mathbb{E}_{(x,T)}\left[\langle F, \mathcal{L}_{(\tau_1,\tau_2)}F\rangle_{\mathcal{H}}\right]. \tag{1.35}$$

Our new proof is much shorter. For example, to derive (R2), integrate (C4) from 0 to T, and take expectations

$$\mathbb{E}_{(x,T)}\left[\int_0^T d|\nabla_\sigma^{||}F_\tau|\,d\tau\right] \overset{(C4)}{\geqslant} \mathbb{E}_{(x,T)}\left[\int_0^T \langle\nabla_\tau|\nabla_\sigma^{||}F_\tau|, dW_\tau\rangle + |\nabla_\sigma^{||}F_\sigma| d\delta_\sigma(\tau)\right]$$
$$\implies \mathbb{E}_{(x,T)}\left[|\nabla_\sigma^{||}F|\right] - \mathbb{E}_{(x,T)}\left[|\nabla_\sigma^{||}F_\sigma|\right] \geqslant 0 \tag{1.36}$$

Then take limits as $\sigma \to 0$ to yield the result

$$|\nabla_x \mathbb{E}_{(x,T)}[F]| = \mathbb{E}_{(x,T)}\left[|\nabla_0^{||}F_0|\right] \leqslant \mathbb{E}_{(x,T)}\left[|\nabla_0^{||}F|\right]. \tag{1.37}$$

Part II is organized as follows:

- In Chapter 2, we shall discuss the geometric and probabilistic preliminaries needed for the proofs of our main theorems.
- In Chapter 3, we shall prove Theorem 1.1, the Bochner formula for martingales on parabolic path space.
- In Chapter 4, we shall discuss the four aforementioned applications of our analysis on path space, i.e. Theorems 1.2, 1.3, 1.4 and 1.5.

2 PRELIMINARIES

2.1 GEOMETRIC PRELIMINARIES

To begin this section, we shall recall the concepts relevant to the construction of the frame bundle on evolving manifolds. An expression of the canonical horizontal (H_a and D_t) and vertical (V_{ab}) vector fields and their commutators will complete this preliminary section.

In time-independent geometry, given a complete Riemannian manifold M, one considers the orthonormal frame bundle $\pi : F \to M$, where the fibres are orthonormal maps $F_x := \{u : \mathbb{R}^n \to T_x M \text{ orthonormal}\}$. To each curve $x_t \in M$, one can associate a horizontal lift $u_t \in F$. In particular, to each vector $X \in T_x M$, given $u \in \pi^{-1}(x)$, one can associate its horizontal lift $X^* \in T_u F$.

We shall now explain, following [Ham93], [HN18a] and [Per08], how these notions can be adapted to the time-dependent setting. To make the appropriate adjustment, we begin by defining space-time $\mathcal{M}$ and the equipped connection ∇ as follows:

Definition 2.1. (Space-time) Let $(M, g_t)_{t\in I}$ be an evolving family of Riemannian manifolds. The space-time is then defined as $\mathcal{M} = M \times I$ equipped with the space-time connection defined on vector fields by $\nabla_X Y = \nabla_X^{g_t} Y$ and $\nabla_t Y = \partial_t Y + \frac{1}{2}\partial_t g_t(Y, \cdot)^{\#g_t}$.

Also observe that this choice of connection is compatible with the metric, namely

$$\frac{d}{dt}\langle X, Y\rangle_{g_t} = \langle \nabla_t X, Y\rangle_{g_t} + \langle X, \nabla_t Y\rangle_{g_t}. \tag{2.1}$$

Generalizing the earlier time-independent construction, we consider the $\mathcal{O}_n$-bundle $\pi : \mathcal{F} \to \mathcal{M}$, where the fibres are given by

$$\mathcal{F}_{(x,t)} := \{u : \mathbb{R}^n \to (T_x M, g_t) \text{ orthonormal}\}. \tag{2.2}$$

To each curve $\gamma_t \in \mathcal{M}$, we can now associate a horizontal lift $u_t \in \mathcal{F}$. Namely, given $u_0 \in \pi^{-1}(\gamma_0)$, the curve u_t is the unique solution of $\pi(u_t) =$

γ_t and $\nabla_{\dot{\gamma}_t}(u_t e_a) = 0$ for $a \in \{1, 2, ..., n\}$, where ∇ is the space-time connection from Definition 2.1. More explicitly, we provide the following formal definition:

Definition 2.2. (Horizontal lift) Given a vector $\alpha X + \beta \partial_t \in T_{(x,t)}\mathcal{M}$ and a frame $u \in \mathcal{F}_{(x,t)}$, there is a unique horizontal lift $\alpha X^* + \beta D_t$ satisfying $\pi_*(\alpha X^* + \beta D_t) = \alpha X + \beta \partial_t$. In particular, X^* is the horizontal lift of $X \in T_x M$ with respect to the fixed metric g_t.

Note that there are $n+1$ canonical horizontal vector fields on $\mathcal{F}$, namely the time-like horizontal vector field D_t defined as the horizontal lift of ∂_t and the space-like horizontal vector fields $\{H_a\}_{a=1}^n$ defined by $H_a(u) = (ue_a)^*$. Also note the notion of vertical vector fields given by $V_{ab}(u) = \frac{d}{d\varepsilon}|_{\varepsilon=0}(u \exp(\varepsilon A_{ab}))$ where $(A_{ab})_{cd} = (\delta_{ac}\delta_{bd} - \delta_{bc}\delta_{ad}) \in M_n(\mathbb{R})$. We now want to express these horizontal and vertical vector fields in local coordinates as follows:

Definition 2.3. (Local coordinates) We view $\mathcal{F}$ as a sub-bundle of the GL_n-bundle $\pi : \mathcal{G} \to \mathcal{M}$ where $\mathcal{G}_{(x,t)} := \{u : \mathbb{R}^n \to (T_x M, g_t) \text{ invertible, linear}\}$. Then, when given local coordinates $(x^1, ..., x^n, t)$ on $\mathcal{M}$, we get local coordinates (x^i, t, e_a^j) on $\mathcal{G}$, where e_a^j is defined by $ue_a = e_a^j \frac{\partial}{\partial x^j}$.

Also note that on $\mathcal{F}$ we have $\delta_{ab} = g(ue_a, ue_b) = g_{ij} e_a^i e_b^j$ and thus we can express the inverse metric as

$$g^{ij} = e_a^i e_a^j. \tag{2.3}$$

It now remains in this section to both write out the canonical vector fields explicitly in local coordinates and derive some commutator relations between them.

Lemma 2.4 (cf. [Ham93]). *In local coordinates, the canonical horizontal vector fields H_a and D_t and canonical vertical vector fields V_{ab} can be expressed as*

$$\begin{cases} H_a &= e_a^j \frac{\partial}{\partial x^j} - e_a^j e_b^k \Gamma_{jk}^\ell \frac{\partial}{\partial e_b^\ell} \\ V_{ab} &= e_b^j \frac{\partial}{\partial e_a^j} - e_a^j \frac{\partial}{\partial e_b^j} \\ D_t &= \partial_t - \frac{1}{2} \widetilde{\partial_t g}_{ab} e_b^\ell \frac{\partial}{\partial e_a^\ell}, \end{cases} \tag{2.4}$$

where $(\widetilde{\partial_t g})_{ab}(u) := (\partial_t g)_{\pi(u)}(ue_a, ue_b)$.

Proof. The canonical horizontal vector fields, H_a, are exactly the same as in [Ham93]. Since the canonical projection $\pi : F(M) \to M$ induces an isomorphism $\pi_* : H_u F(M) \to T_{\pi u} M$, the horizontal lift is uniquely defined by the push-forward $\pi_*(\dot{u}_t) = u_t e_a$ at $t = 0$ with initial condition $u_0 = u$. Then we can write an expression for the horizontal lift $H_a(u)$

$$H_a(u) = (ue_a)^* = (\pi_*(\dot{u}_0))^* = \dot{u}_0 = \dot{x}_0^j \frac{\partial}{\partial x^j} + \dot{e}_c^\ell(0) \frac{\partial}{\partial e_c^\ell}. \tag{2.5}$$

By construction, the vector field $\pi_*(\dot{u}_t) = u_t e_a$ is parallel transported along $x_t = \{x_t^a\}$ for all a, and we can write the following differential equation for $v_a^k(t) = e_a^b(t) X_b^k$

$$\begin{aligned}
\nabla_{\dot{x}_t} \left(e_a^b(t) X_b \right) &= \dot{x}_t^j \nabla_j \left(e_a^b(t) X_b^k e_k \right) \\
&= \dot{x}_t^j e_a^b(t) X_b^k (\nabla_j e_k) + \dot{x}_t^j e_k \nabla_j (e_a^b(t) X_b^k) \\
&= \dot{x}_t^j e_a^b(t) X_b^k \Gamma_{jk}^\ell(x_t) e_\ell + \dot{x}_t^j e_k (\nabla_j v_a^k(t)) \\
&= \dot{x}_t^j v_a^k(t) \Gamma_{jk}^\ell(x_t) e_\ell + \left(\dot{x}_t^j \nabla_j v_a^k(t) \right) e_k \\
&= \left(\Gamma_{jk}^\ell(x_t) \dot{x}_t^j v_a^\ell(t) + \dot{v}_a^k(t) \right) e_\ell \\
&\equiv 0,
\end{aligned} \tag{2.6}$$

and thus, at $t = 0$, we obtain

$$\Gamma_{jl}^k(x_0) \dot{x}_0^j v_m^l(0) + \dot{v}_m^k(0) = \Gamma_{jl}^k(x) e_i^j e_m^l + \dot{e}_m^k(0) = 0.$$

Hence, we obtain

$$H_a(u) = \dot{u}(0) = \dot{x}(0) \frac{\partial}{\partial x^j} + \dot{e}_c^\ell \frac{\partial}{\partial e_c^\ell} = e_a^j \frac{\partial}{\partial x^j} - e_a^j e_b^k \Gamma_{jk}^\ell \frac{\partial}{\partial e_c^\ell} \tag{2.7}$$

as desired.

Next, considering the curve $u(\varepsilon) = u \exp(\varepsilon A_{ab})$, recall that e_c^j and $A_{ab} e_c$ are defined via the relations $ue_c = e_c^j \frac{\partial}{\partial x^j}$ and $A_{ab} e_c = \delta_{ca} e_b - \delta_{cb} e_a$. Then derive

$$\dot{u}(0) e_c = \dot{e}_c^j(0) \frac{\partial}{\partial x^j} = u A_{ab} e_c = \delta_{ac} u e_b - \delta_{bc} u e_a = \left(\delta_{ac} e_b^j - \delta_{bc} e_a^j \right) \frac{\partial}{\partial x^j}, \tag{2.8}$$

whence

$$V_{ab} = \dot{u}(0) = \dot{e}_c^j(0) \frac{\partial}{\partial e_c^j} = \left(\delta_{ac} e_b^j - \delta_{bc} e_a^j \right) \frac{\partial}{\partial e_c^j} = e_b^j \frac{\partial}{\partial e_a^j} - e_a^j \frac{\partial}{\partial e_b^j} \tag{2.9}$$

as given.

Finally we recall that D_t is defined as the horizontal lift of ∂_t. More explicitly, given $u_0 \in \mathcal{F}$, suppose $\pi(u_0) = (x_0, t_0)$ and $\gamma_t := (x_0, t_0 + t)$ and let u_t be the horizontal lift of γ_t. Then, we have that $D_t(u_0) = \frac{d}{dt}|_{t=0} u_t$. Recalling Definition 2.1, and using the tensorial transformation rule $\widetilde{(\partial_t g)}_{ab} = \partial_t g_{jk} e_a^j e_b^k$ (see equation (2.13) below), we compute

$$\begin{aligned}
\nabla_t \left(e_a^j \frac{\partial}{\partial x^j} \right) &= \frac{d(e_a^j)}{dt} \frac{\partial}{\partial x^j} + e_a^j \nabla_t \left(\frac{\partial}{\partial x^j} \right) \\
&= \frac{d(e_a^\ell)}{dt} \frac{\partial}{\partial x^\ell} + \frac{1}{2} e_a^j \partial_t g_{jk} g^{k\ell} \frac{\partial}{\partial x^\ell} \\
&= \frac{d(e_a^\ell)}{dt} \frac{\partial}{\partial x^\ell} + \frac{1}{2} e_a^j \partial_t g_{jk} e_b^k e_b^\ell \frac{\partial}{\partial x^\ell} \\
&= \frac{d(e_a^\ell)}{dt} \frac{\partial}{\partial x^\ell} + \frac{1}{2} \widetilde{\partial_t g}_{ab} e_b^\ell \frac{\partial}{\partial x^\ell}.
\end{aligned} \tag{2.10}$$

It follows that, since $\dot{\gamma}_t = \partial_t$ and $u_t e_a = e_a^\ell(t) \frac{\partial}{\partial x^\ell}$,

$$\nabla_{\dot{\gamma}_t}(u_t e_a) = \nabla_t \left(e_a^\ell(t) \frac{\partial}{\partial x^\ell} \right) = \left(\frac{d}{dt}(e_a^\ell(t)) + \frac{1}{2} \widetilde{\partial_t g}_{ab} e_b^\ell \right) \frac{\partial}{\partial x^\ell} = 0. \tag{2.11}$$

By exhibiting $D_t(u_0)$ in local coordinates

$$D_t(u_0) = 0 \cdot \frac{\partial}{\partial x^j} + 1 \cdot \frac{\partial}{\partial t} + \frac{d}{dt}|_{t=0}(e_a^\ell(t)) \frac{\partial}{\partial e_a^\ell} = \partial_t - \frac{1}{2} \widetilde{\partial_t g}_{ab} e_b^\ell \frac{\partial}{\partial e_a^\ell}, \tag{2.12}$$

we conclude the proof. □

We now recall that the time-dependent tensor fields T correspond to equivariant functions $\widetilde{T}$ on $\mathcal{F}$. For example, a function $f : \mathcal{M} \to \mathbb{R}$ corresponds to the invariant function $\widetilde{f} = f \circ \pi : \mathcal{F} \to \mathbb{R}$ and a time-dependent two-tensor $T = T_{ij}(x,t)\, dx^i \otimes dx^j$ corresponds to an equivariant function $\widetilde{T} = (\widetilde{T}_{ab}) : \mathcal{F} \to \mathbb{R}^{n \times n}$ via $\widetilde{T}_{ab}(u) = T_{\pi(u)}(ue_a, ue_b)$. Note that identities $ue_a = e_a^j \frac{\partial}{\partial x^j}$ and $ue_b = e_b^k \frac{\partial}{\partial x^k}$ yield the transformation rule

$$\widetilde{T}_{ab} = T_{ij} e_a^i e_b^j. \tag{2.13}$$

Also observe that using equations (2.4) and (2.13), one obtains the formula

$$
\begin{aligned}
V_{ab}\tilde{T}_{cd} &= \left(e^k_b \frac{\partial}{\partial e^k_a} - e^k_a \frac{\partial}{\partial e^k_b} \right) (T_{ij} e^i_c e^j_d) \\
&= T_{ij} \left(e^k_b \delta^a_c \delta^i_k e^j_d + e^k_b e^i_c \delta^a_d \delta^j_k e^i_c - e^k_a \delta^b_c \delta^i_k e^j_d - e^k_a e^i_c \delta^b_d \delta^j_k \right) \\
&= T_{ij} \left(e^i_b e^j_d \delta^a_c + e^j_b e^i_c \delta^a_d - e^i_a e^j_d \delta^b_c - e^j_a e^i_c \delta^b_d \right) \\
&= \widetilde{T}_{bd} \delta^a_c - \widetilde{T}_{ad} \delta^b_c + \widetilde{T}_{cb} \delta^a_d - \widetilde{T}_{ca} \delta^b_d.
\end{aligned} \tag{2.14}
$$

Proposition 2.5. (Derivatives) [HN18a] From the correspondence with equivariant functions, the first and second order derivatives of tensor fields can be computed as follows

$$
\begin{cases}
\widetilde{\nabla_X T} = X^* \widetilde{T} \\
\widetilde{\nabla_t T} = D_t \widetilde{T} \\
\widetilde{\Delta T} = \sum_{a=1}^n H_a H_a \widetilde{T} =: \Delta_H \widetilde{T} \\
(\nabla^2 f)(ue_a, ue_b) = H_a H_b \widetilde{f}
\end{cases} \tag{2.15}
$$

Proof. Except for the fourth identity regarding the Hessian, these are either classical results from differential geometry or have already been proven in Lemmas 3.1 and 3.3 of [HN18a]. For this last identity, write the canonical horizontal vector fields in local coordinates and compute

$$
\begin{aligned}
H_a H_b \widetilde{f} &= \left(e^j_a \frac{\partial}{\partial x^j} - e^j_a e^k_c \Gamma^\ell_{jk} \frac{\partial}{\partial e^\ell_c} \right) e^p_b \frac{\partial}{\partial x^p} \widetilde{f} \\
&= e^j_a e^k_b \left(\frac{\partial}{\partial x^j} \frac{\partial}{\partial x^k} \widetilde{f} - \Gamma^p_{jk} \frac{\partial}{\partial x^p} \widetilde{f} \right) \\
&= e^j_a e^k_b \nabla_j \nabla_k f \\
&= \nabla^2 f(ue_a, ue_b),
\end{aligned} \tag{2.16}
$$

thereby proving the proposition. □

Next we proceed to prove a few commutator relations between the newly defined vector field, D_t, and the canonical horizontal and vertical vector fields.

Lemma 2.6 (cf. [Ham93]). *The fundamental vectors fields on the frame bundle satisfy the following commutator relations*

$$\begin{cases} [H_a, H_b] &= \frac{1}{2} R_{abcd} V_{cd} \\ [V_{ab}, H_c] &= \delta_{ac} H_b - \delta_{bc} H_a \\ [V_{ab}, V_{cd}] &= \delta_{bd} V_{ac} - \delta_{ad} V_{bc} + \delta_{ac} V_{bd} - \delta_{bc} V_{ad} \\ [D_t, H_a] &= -\frac{1}{2} \widetilde{\partial_t g}_{ad} H_d + \frac{1}{2} H_b \widetilde{\partial_t g}_{ac} V_{cb} \\ [D_t, V_{ab}] &= 0. \end{cases} \tag{2.17}$$

Proof. We start by deriving the commutator relations not involving D_t from basic differential geometry. Firstly we take the more deliberate approach of breaking the commutator of horizontal lifts into four parts

$$\begin{aligned} [H_a, H_b] = &\left[e_a^j \frac{\partial}{\partial x^j}, e_b^{j'} \frac{\partial}{\partial x^{j'}} \right] - \left[e_a^j \frac{\partial}{\partial x^j}, e_b^{j'} e_d^{k'} \Gamma_{j'k'}^{\ell'} \frac{\partial}{\partial e_d^{\ell'}} \right] \\ &- \left[e_a^j e_c^k \Gamma_{jk}^{\ell} \frac{\partial}{\partial e_c^{\ell}}, e_b^{j'} \frac{\partial}{\partial x^{j'}} \right] + \left[e_a^j e_c^k \Gamma_{jk}^{\ell} \frac{\partial}{\partial e_c^{\ell}}, e_b^{j'} e_d^{k'} \Gamma_{j'k'}^{\ell'}(x) \frac{\partial}{\partial e_d^{\ell'}} \right] \\ =&: \text{(I)} - \text{(II)} - \text{(III)} + \text{(IV)}. \end{aligned} \tag{2.18}$$

It is easy to check

$$\text{(I)} = \left[e_a^j \frac{\partial}{\partial x^j}, e_b^{j'} \frac{\partial}{\partial x^{j'}} \right] = \left(e_a^j e_b^{j'} - e_b^{j'} e_a^j \right) \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^{j'}} = 0. \tag{2.19}$$

Next we calculate

$$\begin{aligned} &\text{(II)} + \text{(III)} \\ &= \left[e_a^j \frac{\partial}{\partial x^j}, e_b^{j'} e_d^{k'} \Gamma_{j'k'}^{\ell'} \frac{\partial}{\partial e_d^{\ell'}} \right] + \left[e_a^j e_c^k \Gamma_{jk}^{\ell} \frac{\partial}{\partial e_c^{\ell}}, e_b^{j'} \frac{\partial}{\partial x^{j'}} \right] \\ &= e_a^j e_b^{j'} e_d^{k'} (\partial_j \Gamma_{j'k'}^{\ell'}) \frac{\partial}{\partial e_d^{\ell'}} - e_b^{j'} e_d^{k'} \delta_a^d \delta_{\ell'}^j \Gamma_{j'k'}^{\ell'} \frac{\partial}{\partial x^j} \\ &\quad - e_a^j e_c^k e_b^{j'} (\partial_{j'} \Gamma_{jk}^{\ell}) \frac{\partial}{\partial e_c^{\ell}} + e_a^j e_c^k \delta_b^c \delta_{\ell}^{j'} \Gamma_{jk}^{\ell} \frac{\partial}{\partial x^{j'}} \\ &= -e_b^{j'} e_a^{k'} \Gamma_{j'k'}^{\ell'} \frac{\partial}{\partial x^{\ell'}} + e_a^j e_b^k \Gamma_{jk}^{\ell} \frac{\partial}{\partial x^{\ell}} + e_a^j e_b^{j'} e_c^k (\partial_j \Gamma_{j'k}^{\ell}) \frac{\partial}{\partial e_c^{\ell}} - e_a^j e_b^{j'} e_c^k (\partial_{j'} \Gamma_{jk}^{\ell}) \frac{\partial}{\partial e_c^{\ell}} \\ &= e_a^j e_b^{j'} e_c^k (\partial_j \Gamma_{j'k}^{\ell} - \partial_{j'} \Gamma_{jk}^{\ell}) \frac{\partial}{\partial e_c^{\ell}}. \end{aligned} \tag{2.20}$$

and

$$
\begin{aligned}
(\mathrm{IV}) &= \left[e_a^j e_c^k \Gamma_{jk}^{\ell} \frac{\partial}{\partial e_c^{\ell}}, e_b^{j'} e_d^{k'} \Gamma_{j'k'}^{\ell'}(x) \frac{\partial}{\partial e_d^{\ell'}} \right] \\
&= e_a^j e_c^k \Gamma_{jk}^{\ell} \Gamma_{j'k'}^{\ell'} \left(e_b^{j'} \delta_d^c \delta_{\ell}^{k'} + e_d^{k'} \delta_b^c \delta_{\ell}^{j'} \right) \frac{\partial}{\partial e_d^{\ell'}} - e_b^{j'} e_d^{k'} \Gamma_{jk}^{\ell} \Gamma_{j'k'}^{\ell'} \left(e_a^j \delta_c^d \delta_{\ell'}^k + e_c^k \delta_a^d \delta_{\ell'}^j \right) \frac{\partial}{\partial e_c^{\ell}} \\
&= e_d^k \Gamma_{jk}^{k'} \Gamma_{j'k'}^{\ell'} (e_a^j e_b^{j'} - e_b^j e_a^{j'}) \frac{\partial}{\partial e_d^{\ell'}} + \left(e_a^j e_b^k e_d^{k'} \Gamma_{jk}^{j'} \Gamma_{j'k'}^{\ell'} - e_b^j e_a^l e_d^{k'} \Gamma_{jk}^{j'} \Gamma_{j'k'}^{\ell'} \right) \frac{\partial}{\partial e_d^{\ell'}} \\
&= e_d^k e_a^j e_b^{j'} \left(\Gamma_{jk}^{k'} \Gamma_{j'k'}^{\ell'} - \Gamma_{j'k}^{k'} \Gamma_{jk'}^{\ell'} \right) \frac{\partial}{\partial e_d^{\ell'}}. \qquad (2.21)
\end{aligned}
$$

Lastly we absorb the four terms to get

$$
\begin{aligned}
[H_a, H_b] &= -e_a^j e_b^{j'} e_c^k (\partial_j \Gamma_{j'k}^{\ell} - \partial_{j'} \Gamma_{jk}^{\ell}) \frac{\partial}{\partial e_c^{\ell}} + e_d^k e_a^j e_b^{j'} \left(\Gamma_{jk}^{k'} \Gamma_{j'k'}^{\ell'} - \Gamma_{j'k}^{k'} \Gamma_{jk'}^{\ell'} \right) \frac{\partial}{\partial e_d^{\ell'}} \\
&= e_a^j e_b^{j'} e_d^k \left(\partial_{j'} \Gamma_{jk}^{\ell} - \partial_j \Gamma_{j'k}^{\ell} + \Gamma_{jk}^{k'} \Gamma_{j'k'}^{\ell'} - \Gamma_{j'k}^{k'} \Gamma_{jk'}^{\ell'} \right) \frac{\partial}{\partial e_d^{\ell}} \\
&= e_a^j e_b^{j'} e_d^k R_{j'jk}^{\ell} \frac{\partial}{\partial e_d^{\ell}} \\
&= \frac{1}{2} R_{abcd} V_{cd}. \qquad (2.22)
\end{aligned}
$$

Next, we compute the second commutator

$$
\begin{aligned}
&[V_{ab}, H_c] \\
&= \left[e_b^j \frac{\partial}{\partial e_a^j} - e_a^j \frac{\partial}{\partial e_b^j}, e_c^{j'} \frac{\partial}{\partial x^{j'}} - e_c^{j'} e_d^{k'} \Gamma_{j'k'}^{\ell'} \frac{\partial}{\partial e_d^{\ell'}} \right] \\
&= e_b^j \delta_{ac} \left(\frac{\partial}{\partial x^j} - e_d^{k'} \Gamma_{jk'}^{\ell'} \frac{\partial}{\partial e_d^{\ell'}} \right) - e_b^j e_c^{j'} \delta_d^a \delta_j^{k'} \Gamma_{j'k'}^{\ell'} \frac{\partial}{\partial e_d^{\ell'}} + e_c^{j'} e_d^{k'} \Gamma_{j'k'}^{\ell'} \delta_b^d \delta_{\ell'}^j \frac{\partial}{\partial e_a^j} \\
&\quad - e_a^j \delta_{bc} \left(\frac{\partial}{\partial x^j} - e_d^{k'} \Gamma_{jk'}^{\ell'} \frac{\partial}{\partial e_d^{\ell'}} \right) + e_a^j e_c^{j'} \delta_d^b \delta_j^{k'} \Gamma_{j'k'}^{\ell'} \frac{\partial}{\partial e_d^{\ell'}} - e_c^{j'} e_d^{k'} \Gamma_{j'k'}^{\ell'} \delta_a^d \delta_{\ell'}^j \frac{\partial}{\partial e_b^j} \\
&= \delta_{ac} H_b - \delta_{bc} H_a - e_c^{j'} \Gamma_{j'j}^{\ell'} \left(e_b^j \frac{\partial}{\partial e_a^{\ell'}} - e_a^j \frac{\partial}{\partial e_b^{\ell'}} \right) + e_c^{j'} \Gamma_{j'k'}^{j} \left(e_b^{k'} \frac{\partial}{\partial e_a^j} - e_a^{k'} \frac{\partial}{\partial e_b^j} \right) \\
&= \delta_{ac} H_b - \delta_{bc} H_a, \qquad (2.23)
\end{aligned}
$$

and finally,

$$
\begin{aligned}
[V_{ab}, V_{cd}] &= \left[e_b^j \frac{\partial}{\partial e_a^j} - e_a^j \frac{\partial}{\partial e_b^j}, e_d^{j'} \frac{\partial}{\partial e_c^{j'}} - e_c^{j'} \frac{\partial}{\partial e_d^{j'}} \right] \\
&= e_b^j \delta_{ad} \frac{\partial}{\partial e_c^j} - e_d^j \delta_{bc} \frac{\partial}{\partial e_a^j} - e_b^j \delta_{ac} \frac{\partial}{\partial e_d^j} + e_c^j \delta_{bd} \frac{\partial}{\partial e_a^j} \\
&\quad - e_a^j \delta_{bd} \frac{\partial}{\partial e_c^j} + e_d^j \delta_{ac} \frac{\partial}{\partial e_b^j} + e_a^j \delta_{bc} \frac{\partial}{\partial e_d^j} - e_c^j \delta_{ad} \frac{\partial}{\partial e_b^j} \\
&= \delta_{bd} V_{ac} - \delta_{ad} V_{bc} + \delta_{ac} V_{bd} - \delta_{bc} V_{ad}. \qquad (2.24)
\end{aligned}
$$

It remains to check the final two commutator relations. To prove the first of these, between D_t and the horizontal vector field H_a, we compute

$$
\begin{aligned}
[\partial_t, H_a] &= \left[\partial_t, e_a^j \frac{\partial}{\partial x^j} - e_a^j e_b^k \Gamma_{jk}^{\ell} \frac{\partial}{\partial e_b^{\ell}} \right] \\
&= -e_a^j e_b^k \partial_t \Gamma_{jk}^{\ell} \frac{\partial}{\partial e_b^{\ell}} \\
&= -\frac{1}{2} e_a^j e_b^k (g^{\ell p} (\nabla_j (\partial_t g_{kp}) + \nabla_k (\partial_t g_{jp}) - \nabla_p (\partial_t g_{jk})) \frac{\partial}{\partial e_b^{\ell}} \\
&= -\frac{1}{2} e_a^j e_b^k e_c^{\ell} e_c^p \left(\nabla_j (\partial_t g_{kp}) + \nabla_k (\partial_t g_{jp}) - \nabla_p (\partial_t g_{jk}) \right) \frac{\partial}{\partial e_b^{\ell}} \\
&= -\frac{1}{2} e_c^{\ell} \left((\widetilde{\nabla \partial_t g})_{abc} + (\widetilde{\nabla \partial_t g})_{bac} - (\widetilde{\nabla \partial_t g})_{cab} \right) \frac{\partial}{\partial e_b^{\ell}} \\
&= -\frac{1}{2} e_c^{\ell} (H_a \widetilde{\partial_t g}_{bc} + H_b \widetilde{\partial_t g}_{ac} - H_c \widetilde{\partial_t g}_{ab}) \frac{\partial}{\partial e_b^{\ell}} \\
&= -\frac{1}{2} H_a (\widetilde{\partial_t g}_{bc}) e_c^{\ell} \frac{\partial}{\partial e_b^{\ell}} + \frac{1}{2} H_b (\widetilde{\partial_t g}_{ac}) \left(e_b^{\ell} \frac{\partial}{\partial e_c^{\ell}} - e_c^{\ell} \frac{\partial}{\partial e_b^{\ell}} \right) \\
&= -\frac{1}{2} H_a (\widetilde{\partial_t g}_{bc}) e_c^{\ell} \frac{\partial}{\partial e_b^{\ell}} + \frac{1}{2} H_b (\widetilde{\partial_t g}_{ac}) V_{cb} \qquad (2.25)
\end{aligned}
$$

and

$$
\begin{aligned}
[D_t - \partial_t, H_a] &= [-\frac{1}{2}(\widetilde{\partial_t g_{cd}})e_d^{\ell'}\frac{\partial}{\partial e_c^{\ell'}}, H_a] \\
&= -\frac{1}{2}\widetilde{\partial_t g_{cd}}\left[e_d^{\ell'}\frac{\partial}{\partial e_c^{\ell'}}, H_a\right] + \frac{1}{2}H_a(\widetilde{\partial_t g_{cd}})e_d^{\ell'}\frac{\partial}{\partial e_c^{\ell'}} \\
&= -\frac{1}{2}\widetilde{\partial_t g_{cd}}\left(\delta_a^c e_d^j\frac{\partial}{\partial x^j} - (\delta_a^c e_d^j e_b^k + \delta_b^c e_a^j e_d^k)\Gamma_{jk}^\ell\frac{\partial}{\partial e_b^\ell}\right) \\
&\quad - \frac{1}{2}\widetilde{\partial_t g_{cd}}\delta_d^b e_a^j e_b^k\Gamma_{jk}^\ell\frac{\partial}{\partial e_c^\ell} + \frac{1}{2}H_a(\widetilde{\partial_t g_{bc}})e_c^\ell\frac{\partial}{\partial e_b^\ell} \\
&= -\frac{1}{2}\widetilde{\partial_t g_{cd}}\delta_a^c H_d + \frac{1}{2}H_a(\widetilde{\partial_t g_{bc}})e_c^\ell\frac{\partial}{\partial e_b^\ell} \\
&= -\frac{1}{2}\widetilde{\partial_t g_{ad}}H_d + \frac{1}{2}H_a(\widetilde{\partial_t g_{bc}})e_c^\ell\frac{\partial}{\partial e_b^\ell}.
\end{aligned}
\tag{2.26}
$$

Next, we sum equations (2.25) and (2.26) to compute the desired commutator relation

$$
[D_t, H_a] = [\partial_t, H_a] + [D_t - \partial_t, H_a] = -\frac{1}{2}\widetilde{\partial_t g_{ad}}H_d + \frac{1}{2}H_b(\widetilde{\partial_t g_{ac}})V_{cb}. \tag{2.27}
$$

Finally, using equations (2.4) and (2.14), the commutator of D_t and V_{ab} is

$$
\begin{aligned}
[D_t, V_{ab}] &= \left[\partial_t - \frac{1}{2}(\widetilde{\partial_t g_{cd}})e_d^{\ell'}\frac{\partial}{\partial e_c^{\ell'}}, V_{ab}\right] \\
&= -\frac{1}{2}\widetilde{\partial_t g_{cd}}\left[e_d^{\ell'}\frac{\partial}{\partial e_c^{\ell'}}, V_{ab}\right] + \frac{1}{2}V_{ab}(\widetilde{\partial_t g_{cd}})e_d^{\ell'}\frac{\partial}{\partial e_c^{\ell'}} \\
&= \frac{1}{2}\widetilde{\partial_t g_{cd}}\left(e_d^{\ell'}\frac{\partial}{\partial e_b^{\ell'}}\delta_c^a + e_b^{\ell'}\frac{\partial}{\partial e_c^{\ell'}}\delta_d^a - e_d^{\ell'}\frac{\partial}{\partial e_a^{\ell'}}\delta_c^b - e_a^{\ell'}\frac{\partial}{\partial e_c^{\ell'}}\delta_d^b\right) \\
&\quad + \frac{1}{2}\left(\widetilde{\partial_t g_{bd}}\delta_c^a + \widetilde{\partial_t g_{cb}}\delta_d^a - \widetilde{\partial_t g_{ad}}\delta_c^b - \widetilde{\partial_t g_{ca}}\delta_d^b\right)e_d^{\ell'}\frac{\partial}{\partial e_c^{\ell'}} \\
&\equiv 0,
\end{aligned}
\tag{2.28}
$$

thereby proving this lemma on commuting canonical vector fields. □

Corollary 2.7. *If $\tilde{f} : \mathcal{F} \to \mathbb{R}$ is an orthonormally invariant function, then*

$$
\begin{cases}
H_a H_b \tilde{f} - H_b H_a \tilde{f} &= 0 \\
\Delta_H H_a \tilde{f} - H_a \Delta_H \tilde{f} &= \widetilde{\mathrm{Rc}}_{ab} H_b \tilde{f},
\end{cases}
\tag{2.29}
$$

where $\widetilde{\mathrm{Rc}}_{ab}(u) = \mathrm{Rc}_{\pi(u)}(ue_a, ue_b)$.

Proof. This is a direct application of the commutator relations from Lemma 2.6.

Since $\tilde{f}$ is orthonormally-invariant, it is constant along fibres and $V_{cd}\tilde{f} \equiv 0$, whence

$$(H_a H_b f - H_b H_a)\,\tilde{f} = [H_a, H_b]\tilde{f} = \frac{1}{2} R_{abcd} V_{cd} \tilde{f} = 0. \tag{2.30}$$

Next we compute

$$\begin{aligned}
\Delta_H H_a \tilde{f} - H_a \Delta_H \tilde{f} &= (H_b H_b H_a - H_a H_b H_b)\,\tilde{f} \\
&= -[H_a, H_b] H_b \tilde{f} \\
&= -\frac{1}{2} R_{abcd} V_{cd} H_b \tilde{f} \quad \text{(by Lemma 2.6)} \\
&= \frac{1}{2} R_{abcd} \left(-[V_{cd}, H_b]\tilde{f} - H_b(V_{cd}\tilde{f}) \right) \\
&= \widetilde{\mathrm{Rc}}_{ab} H_b \tilde{f},
\end{aligned} \tag{2.31}$$

which completes the proof. □

Proposition 2.8. Let $\tilde{f} : \mathcal{F} \to \mathbb{R}$ be an orthonormally invariant function. Then

$$[D_t - \Delta_H, H_a]\tilde{f} = -\frac{1}{2}(\widetilde{\partial_t g} + 2\widetilde{\mathrm{Rc}})_{ab} H_b \tilde{f}. \tag{2.32}$$

Proof. It readily follows from Lemma 2.6 and Corollary 2.7 that

$$\begin{aligned}
[D_t - \Delta_H, H_a]\tilde{f} &= [D_t, H_a]\tilde{f} - [\Delta_H, H_a]\tilde{f} \\
&= -\frac{1}{2}\widetilde{\partial_t g}_{ad} H_d \tilde{f} + \frac{1}{2} H_b(\widetilde{\partial_t g}_{ac}) V_{cb}\tilde{f} - \widetilde{\mathrm{Rc}}_{ab} H_b \tilde{f} \\
&= -\frac{1}{2}(\widetilde{\partial_t g} + 2\widetilde{\mathrm{Rc}})_{ab} H_b \tilde{f},
\end{aligned} \tag{2.33}$$

thereby proving the proposition. □

2.2 PROBABILISTIC PRELIMINARIES

The principal goal of this section is to recall the notions of Brownian motion and stochastic parallel transport in the setting of evolving manifolds as developed in [ACT08] and [HN18a].

We first remark that it shall hereafter be assumed that in addition to the

Riemannian manifolds $\{M_t\}$ being complete as in the previous section, they will also satisfy

$$\sup_{\mathcal{M}} (|\mathrm{Rm}| + |\partial_t g| + |\nabla \partial_t g|) < \infty. \tag{2.34}$$

Horizontal curves $\{u_\tau\}_{\tau\in[0,T]} \in \mathcal{F}$, where $\pi(u_\tau) = (x_\tau, T-\tau)$, correspond to curves $\{w_\tau\}_{\tau\in[0,T]} \in \mathbb{R}^n$ (also known as the anti-development of u_τ) via the following initial value problem

$$\begin{cases} \frac{du_\tau}{d\tau} & = D_\tau + H_a(u_\tau)\frac{dw^a_\tau}{d\tau} \\ w_0 & = 0. \end{cases} \tag{2.35}$$

This definition of the anti-development in the time-dependent geometry setting appropriately motivates the following stochastic differential equation in the case of evolving manifolds

$$\begin{cases} dU_\tau & = D_\tau\, d\tau + H_a(U_\tau) \circ dW^a_\tau \\ U_0 & = u. \end{cases} \tag{2.36}$$

We make a short note on notation that $W_\tau \sim \sqrt{2}B_\tau$ refers to the Brownian motion in $\mathbb{R}^n$ with rescaling by a factor of $\sqrt{2}$ such that it has quadratic variation $d[W,W]_\tau = 2d[B,B]_\tau = 2\,d\tau$ and $\circ$ refers to the Stratonovich integral in differential notation.

Next, by demonstrating that this equation satisfies existence and uniqueness criterion as well as Itô's lemma, the notions of Brownian motion, via projection onto $\mathcal{M}$, and stochastic parallel transport can be formalized.

Proposition 2.9 (cf. [HN18a]). (Existence, uniqueness and Itô's lemma) The stochastic differential equation (2.36) has a unique solution $\{U_\tau\}_{\tau\in[0,T]}$ that satisfies $\pi_2(U_\tau) = T-\tau$ and explosion time $e(U) = \infty$. Moreover, $\tau \mapsto U_\tau(\omega)$ is continuous for every Brownian path $\omega \in C([0,T], \mathbb{R}^n)$, and, given $\tilde{f} : \mathcal{F} \to \mathbb{R}$ is of class C^2, then the solution U_τ satisfies

$$d\tilde{f}(U_\tau) = D_\tau \tilde{f}(U_\tau)\, d\tau + \langle (H\tilde{f})(U_\tau), dW_\tau\rangle + \Delta_H(\tilde{f})(U_\tau)\, d\tau. \tag{2.37}$$

Proof. By Nash's embedding theorem, we embed $\mathcal{F}$ isometrically into $\mathbb{R}^N$ for $N \gg \frac{n^2+n}{2}$ and construct a suitable smooth extension of compact support by partitions of unity. Then there exists a unique solution of system

$$\begin{cases} D_t\, dt + H_j(U_t) \circ dW^j_t \\ U_0 = u_0 \end{cases} \tag{2.38}$$

on $\mathbb{R}^N$. By Theorem 1.2.8 of [Hsu02], using Grönwall's lemma, the extended solution of the stochastic differential equation on $\mathbb{R}^N$ stays inside $\mathcal{F}$ up to explosion time $e(U)$. Moreover, by Theorem 1.2.9 of [Hsu02], this solution is in fact a unique solution on $\mathcal{F}$ up to $e(U)$. Also, as n-dimensional Brownian motion is continuous in τ for almost every path ω, so it is for U_τ.

We now proceed by computing both dU_t and the quadratic variation $d[H(U_t), W_t]_t$

$$\begin{aligned} dU_t^a &= D_t^a\, dt + H_j^a(U_t)\, dW_t^j + \frac{1}{2} d[H_j^a(U), W^j]_t \\ &= D_t^a\, dt + H_j^a(U_t)\, dW_t^j + \frac{1}{2}\partial_b H_j^a(U_t)\, d[U^b, W^j]_t \\ &= D_t^a\, dt + H_j^a(U_t)\, dW_t^j + \frac{1}{2}\partial_b H_j^a(U_t) H_j^b\, d[W^j, W^j]_t \\ &= D_t^a\, dt + H_j^a(U_t)\, dW_t^j + \partial_b H_j^a(U_t) H_j^b\, dt. \end{aligned} \tag{2.39}$$

We are now prepared to derive the desired identity using Itô calculus in $\mathbb{R}^n$

$$\begin{aligned} &d(\tilde{f}(U_t)) \\ &\quad= \langle \nabla \tilde{f}(U_t), dU_t\rangle + \frac{1}{2} d[\tilde{f}(U_t), \tilde{f}(U_t)]_t \\ &\quad= \langle \nabla \tilde{f}(U_t), D_t\, dt + H_j(U_t)\, dW_t^j + \partial_b H_j(U_t) H_j^b\, dt\rangle + \frac{1}{2}\partial^2_{ba}\tilde{f}(U_t) d[U^b, U^a]_t \\ &\quad= \partial_a \tilde{f}(U_t) D_t^a\, dt + H_j(\tilde{f})(U_t)\, dW_t^j \\ &\qquad + \left(\partial_a \tilde{f}(U_t)\partial_b H_j^a(U_t) H_j^b(U_t) + \partial^2_{ba}\tilde{f}(U_t) H_j^b(U_t) H_j^a(U_t)\right) dt \\ &\quad= D_t(\tilde{f})(U_t)\, dt + \langle H(\tilde{f})(U_t), dW_t\rangle \\ &\qquad + \left((H_j^b \partial_b H_j)(\tilde{f}) + (H_j^b H_j)(\partial_b \tilde{f})\right)(U_t)\, dt \\ &\quad= D_t(\tilde{f})(U_t)\, dt + \langle H(\tilde{f})(U_t), dW_t\rangle + H_j^b \partial_b (H_j \tilde{f})(U_t)\, dt \\ &\quad= D_t(\tilde{f})(U_t)\, dt + \langle H(\tilde{f})(U_t), dW_t\rangle + \Delta_H(\tilde{f})(U_t)\, dt. \end{aligned} \tag{2.40}$$

By our convention (1.1), it follows that there exists a distance-like function $r : \mathcal{M} \to \mathbb{R}$ with an extension $\tilde{r} : \mathcal{F} \to \mathbb{R}$ that is independent of both time and fibre coordinates. By (2.37), it follows that the solution to (2.38) has infinite explosion time $e(U) = \infty$ and so there exists a unique solution that always stays inside $\mathcal{F}$. Finally, applying (2.9) to $\tilde{f} = \pi_2$, one gets $d\pi_2(U_\tau) = -d\tau$. Combined with $\pi_2(U_0) = T$, it follows that $\pi_2(U_\tau) = T - \tau$. □

We shall now continue with defining the notions of Brownian motion and stochastic parallel transport, from [HN18a], in the setting of time-evolving families of Riemannian manifolds.

Definition 2.10. (Brownian motion on space time) We call $\pi(U_\tau) = (X_\tau, T - \tau)$ the Brownian motion on space time $\mathcal{M} = M \times I$ with base point $\pi(u) = (x, T)$.

Definition 2.11. (Stochastic parallel transport) The family of isometries

$$\left\{P_\tau = U_0 U_\tau^{-1} : (T_{X_\tau} M, g_{T-\tau}) \to (T_x M, g_T)\right\} \tag{2.41}$$

is called the stochastic parallel transport along the Brownian curve X_τ.

This Brownian motion now inherits a path based space, diffusion measure and filtration. First, we denote by $P_0\mathbb{R}^n$ the based path space on $\mathbb{R}^n$, namely the space of continuous curves $\{w_\tau | w_0 = 0\}_{\tau\in[0,T]} \subset \mathbb{R}^n$.

Definition 2.12. (Based path spaces) Let $P_u\mathcal{F}$ and $P_{(x,T)}\mathcal{M}$ be the based path spaces of continuous curves,

$$P_u\mathcal{F} := \{u_\tau | u_0 = u, \pi_2(u_\tau) = T - \tau\}_{\tau\in[0,T]} \subset \mathcal{F} \tag{2.42}$$

and

$$P_{(x,T)}\mathcal{M} = \{\gamma_\tau = (x_\tau, T - \tau) | \gamma_0 = (x, T)\}_{\tau\in[0,T]}, \tag{2.43}$$

respectively.

To construct the Wiener measure, we first observe that solving the stochastic differential equation (2.36) yields a map $U : P_0\mathbb{R}^n \to P_u\mathcal{F}$. Moreover, the projection map $\pi : \mathcal{F} \to \mathcal{M}$ induces a map $\Pi : P_u\mathcal{F} \to P_{(x,T)}\mathcal{M}$.

Definition 2.13. (Wiener measure) Let $\mathbb{P}_0$ be the Wiener measure on path space $P_0\mathbb{R}^n$. We then say that $\mathbb{P}_u := U_*(\mathbb{P}_0)$ and $\mathbb{P}_{(x,T)} := \Pi_*\mathbb{P}_u$ are the Wiener measures of horizontal Brownian motion on $\mathcal{F}$ and Brownian motion on space-time $\mathcal{M}$ respectively.

Moreover, we can uniquely characterize the Wiener measure in terms of the heat kernel.

Proposition 2.14. [HN18a] Let $\{\tau_j\}_{j=1}^k$ be a partition of $[0, T]$, $U_j \subseteq M$ and $\gamma_0 = (x, T)$. Then

$$\begin{aligned} &\mathbb{P}_{(x,T)}\left[X_{\tau_j} \in U_j,\ \forall j \in 1, ..., k\right] \\ &\quad = \int_{\times_1^k U_j} \Pi_j H(x_{j-1}, T - \tau_{j-1} | x_j, T - \tau_j) \otimes d\mathrm{Vol}_{g_{T-\tau_j}}(y_j) \end{aligned} \tag{2.44}$$

uniquely characterizes the Wiener measure on $P_{(x,T)}\mathcal{M}$.

Proof. The proof follows as in Proposition 3.31 of [HN18a]. □

Next, we recall that the path space $P_0\mathbb{R}^n$ comes equipped with an intrinsic filtration $\Sigma_\tau^{\mathbb{R}^n}$ generated by evaluation maps $\{e_\sigma : P_0\mathbb{R}^n \to \mathbb{R}^n | e_\sigma(w) = w_\sigma\,, \sigma \leqslant \tau\}$.

Definition 2.15. (Filtrations on $P_u\mathcal{F}$ and $P_{(x,T)}\mathcal{M}$) The filtrations on $P_u\mathcal{F}$ and $P_{(x,T)}\mathcal{M}$ are simply the respective push-forwards $\Sigma_\tau^{\mathcal{M}} := (\Pi \circ U)_* \Sigma_\tau^{\mathbb{R}^n}$ and $\Sigma_\tau^{\mathcal{F}} := U_* \Sigma_\tau^{\mathbb{R}^n}$.

A short reiteration of induced martingales as well as parallel and Malliavin gradients are constructed in the time-dependent setting will now complete this section.

Definition 2.16. (Induced martingale) Let $F : P_{(x,T)}\mathcal{M} \to \mathbb{R}$ be integrable. Then, we define the induced martingale as $F_\tau(\gamma) := \mathbb{E}[F|\Sigma_\tau](\gamma)$.

Using this definition, the conditional expectation can now be characterized by a representation formula.

Proposition 2.17. (Conditional expectation) [HN18a] Suppose the conditional expectation is as defined. Then, for almost every Brownian curve $\{\gamma_\tau\}_{\tau\in[0,T]}$,

$$F_\tau(\gamma) := \mathbb{E}[F|\Sigma_\tau](\gamma) = \int_{P_{\gamma_\tau}\mathcal{M}} F(\gamma|_{[0,\tau]} * \gamma') d\mathbb{P}_{\gamma_\tau}(\gamma'), \tag{2.45}$$

where we integrate over all Brownian curves γ' in the based path space $P_{\gamma_\tau}\mathcal{M}$ with respect to Wiener measure $\mathbb{P}_{\gamma_\tau}$ and $*$ denotes concatenation of the two curves $\gamma|_{[0,\tau]}$ and γ'.

Proof. The proof follows as in Proposition 3.19 of [HN18a]. □

To define the two notions of gradients, we first recall that cylinder functions are of the form $u \circ e_{\boldsymbol{\sigma}}$, where $e_{\boldsymbol{\sigma}} : P_{(x,T)}\mathcal{M} \to M^k$ are k-point evaluation maps, namely $e_{\boldsymbol{\sigma}} : \gamma \mapsto (\pi_1\gamma_{\sigma_1}, ..., \pi_1\gamma_{\sigma_k})$, and $u : M^k \to \mathbb{R}$ is compactly supported.

Example 2.18. Let $F(\gamma) := f \circ e_{\tau_1}(\gamma) = f(\pi_1\gamma_{\tau_1})$. Then the induced martingale of F is given for $\tau > \tau_1$ by

$$\begin{aligned} F_\tau(\gamma) &= \int_{P_{\gamma_\tau}\mathcal{M}} F(\gamma|_{[0,\tau]} * \gamma')\, d\mathbb{P}_{\gamma_\tau}(\gamma') \\ &= \int_{P_{\gamma_\tau}\mathcal{M}} f(\pi_1\gamma_{\tau_1})\, d\mathbb{P}_{\gamma_\tau}(\gamma') \\ &= f(X_{\tau_1}), \end{aligned} \tag{2.46}$$

and for $\tau < \tau_1$ by

$$\begin{aligned} F_\tau(\gamma) &= \int_{P_{\gamma_\tau}\mathcal{M}} F(\gamma|_{[0,\tau]} * \gamma')\, d\mathbb{P}_{\gamma_\tau}(\gamma') \\ &= \int_{P_{\gamma_\tau}\mathcal{M}} f(\pi_1 \gamma'_{\tau_1-\tau})\, d\mathbb{P}_{\gamma_\tau}(\gamma') \\ &= \int_M f(y) H(X_\tau, T-\tau | y, T-\tau_1)\, dV_{g_{T-\tau_1}}(y) \\ &= H_{T-\tau_1, T-\tau} f(\pi_1 \gamma_\tau). \end{aligned} \tag{2.47}$$

Definition 2.19. (Parallel gradient) Let $\sigma \in [0,T]$ and let $F : P_{(x,T)}\mathcal{M} \to \mathbb{R}$ be a cylinder function. Then the σ-parallel gradient is the almost everywhere uniquely defined function $\nabla^{||}_\sigma F : P_{(x,T)}\mathcal{M} \to (T_xM, g_T)$ such that

$$D_{V^\sigma} F(\gamma) = \langle \nabla^{||}_\sigma F(\gamma), v \rangle_{(T_xM, g_T)}, \tag{2.48}$$

for almost every Brownian curve γ and $v \in (T_xM, g_T)$, where $V^\sigma_\tau = P_\tau^{-1} v \mathbb{1}_{[\sigma,T]}(\tau)$. Here, D_V denotes the Fréchet derivative.

Example 2.20. Let $F = u \circ e_{\boldsymbol{\tau}}$ be a k-point function with partition $\{\tau_j\}_{j=1}^k$. Then the parallel gradient of F is given by

$$\nabla^{||}_\sigma F = e^*_{\boldsymbol{\tau}} \left(\sum_{\tau_j \geq \sigma} P_{\tau_j} \mathrm{grad}^{(j)}_{g_{T-\tau_j}} u \right). \tag{2.49}$$

Finally, we let $\mathcal{H}$ be the Hilbert space of $W^{1,2}$ curves in (T_xM, g_T) with $v_0 = 0$ equipped with the natural Sobolev inner product given by

$$\langle u, v \rangle_{\mathcal{H}} := \int_0^T \langle \dot{u}_\tau, \dot{v}_\tau \rangle_{(T_xM, g_T)}\, d\tau. \tag{2.50}$$

Definition 2.21. (Malliavin gradient) Let $F : P_{(x,T)}\mathcal{M} \to \mathbb{R}$ be a cylinder function. The Malliavin gradient is the almost everywhere uniquely defined function $\nabla^{\mathcal{H}} F : P_{(x,T)}\mathcal{M} \to \mathcal{H}$ such that

$$D_V F(\gamma) = \langle \nabla^{\mathcal{H}} F(\gamma), v \rangle_{\mathcal{H}}, \tag{2.51}$$

for almost every Brownian curve γ and $v \in \mathcal{H}$, where $V_\tau = P_\tau^{-1} v_\tau$.

Definition 2.22. (Skorokhod integral) The adjoint of the Malliavin gradient, also known as the Skorokhod integral, is the uniquely defined operator $(\nabla^{\mathcal{H}})^* : L^2(P\mathcal{M}) \to L^2(P\mathcal{M})$ such that

$$\mathbb{E}[F(\nabla^{\mathcal{H}})^* g] = \mathbb{E}\left[\langle \nabla^{\mathcal{H}} F, g\rangle_{\mathcal{H}}\right], \tag{2.52}$$

for all $F, g \in L^2(P\mathcal{M})$.

Definition 2.23. (Ornstein-Uhlenbeck operator) The Ornstein-Uhlenbeck operator is defined as

$$\mathcal{L}_{(\tau_1,\tau_2)} := (\nabla^{\mathcal{H}})^* \nabla^{\mathcal{H}} \tag{2.53}$$

where $\nabla^{\mathcal{H}}$ and $(\nabla^{\mathcal{H}})^*$ are the Malliavin gradient and its adjoint from (2.51) and (2.52) respectively.

3

BOCHNER FORMULA ON PARABOLIC PATH SPACE

For convenience of the reader, we shall first recall and prove the statement of Bochner's formula in the time-dependent setting.

Lemma 3.1. *(Bochner) First let $(M, g_t)_{t\in I}$ be a family of Riemannian manifolds and* $\mathrm{grad}^i_{g_t} = g^{ij}_t \partial_j$ *where Δ_{g_t} be the Laplace-Beltrami operator. Then the evolution of $|\nabla u|^2_{g_t}$ is given by*

$$\begin{aligned}
&\frac{1}{2}(-\partial_t + \Delta_{g_t})(|\nabla u|^2_{g_t}) \\
&\quad = \langle \nabla u, \nabla(-\partial_t + \Delta_{g_t})u\rangle + |\nabla^2 u|^2 + \frac{1}{2}(\partial_t g_t + 2\mathrm{Rc}_{g_t})(\mathrm{grad}\, u, \mathrm{grad}\, u).
\end{aligned} \tag{3.1}$$

Proof. We evaluate both

$$\begin{aligned}
\frac{1}{2}\Delta_{g_t}(|\nabla u|^2_{g_t}) &= \frac{1}{2}\nabla_i\nabla_i(\nabla_j u \nabla_j u) \\
&= \left(\nabla_i \nabla_j u\right)\left(\nabla_i \nabla_j u\right) + \left(\nabla_j u\right)\left(\nabla_i\nabla_i\nabla_j u\right) \\
&= |\nabla^2 u|^2 + (\nabla_j u)(\nabla_j \Delta_{g_t} u) + \mathrm{Rc}_{g_t}(\mathrm{grad}\, u, \mathrm{grad}\, u)
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
\frac{1}{2}\partial_t(|\nabla u|^2_{g_t}) &= \frac{1}{2}\partial_t g^{ij}_t \nabla_i u \nabla_j u + g^{ij}_t \nabla_i u \partial_t(\nabla_j u) \\
&= -\frac{1}{2}\partial_t g_{k\ell} g^{ki} g^{\ell j}\nabla_i u \nabla_j u + g^{ij}_t \nabla_i u \partial_t(\nabla_j u) \\
&= \langle \nabla u, \nabla(\partial_t u)\rangle - \frac{1}{2}\partial_t g_t(\mathrm{grad}\, u, \mathrm{grad}\, u).
\end{aligned} \tag{3.3}$$

We then deduce the Bochner formula as the difference of the two results. □

Theorem 3.2. *(Martingale representation theorem) If $F_\tau : P_{(x,T)}\mathcal{M} \to \mathbb{R}$ is a martingale on parabolic path space and $F_\tau \in \mathcal{D}(\nabla^{||}_\tau)$, then F_τ solves stochastic differential equation*

$$\begin{cases} dF_\tau = \langle \nabla^{||}_\tau F_\tau, dW_\tau\rangle \\ F|_{\tau=0} = F_0. \end{cases} \tag{3.4}$$

Proof. By approximation (cf. [HN18a, Sec 2.4]), it suffices to prove the theorem in the case where F_τ is a martingale induced by a k-point cylinder function. Namely, let $F(\gamma) = f(\pi_1\gamma_{\tau_1}, ..., \pi_1\gamma_{\tau_k})$, where $f : M^k \to \mathbb{R}$ and we recall that $\gamma_\tau = (X_\tau, T-\tau)$. Also let $F_\tau = \mathbb{E}_{(x,T)}[F|\Sigma_\tau]$ be the induced martingale. Then, for $\tau \in (\tau_\ell, \tau_{\ell+1})$ by Propositions 2.17 and then 2.14, we calculate

$$\begin{aligned}
F_\tau(\gamma) &= \int_{P_{\gamma_\tau}\mathcal{M}} F(\gamma|_{[0,\tau]} * \gamma')\, d\mathbb{P}_{\gamma_\tau}(\gamma') \\
&= \int_{P_{\gamma_\tau}\mathcal{M}} f(\pi_1\gamma_{\tau_1}, ..., \pi_1\gamma_{\tau_\ell}, \pi_1\gamma'_{\tau_{\ell+1}-\tau}, ..., \pi_1\gamma'_{\tau_k-\tau})\, d\mathbb{P}_{\gamma_\tau}(\gamma') \\
&= \int_{M^{k-\ell}} f(X_{\tau_1}, \dots, X_{\tau_\ell}, y_{\ell+1}, \dots, y_k) H(X_\tau, T-\tau | y_{\ell+1}, T-\tau_{\ell+1}) \\
&\quad H(y_{\ell+1}, T-\tau_{\ell+1}|y_{\ell+2}, T-\tau_{\ell+2}) \cdots H(y_{k-1}, T-\tau_{k-1}|y_k, T-\tau_k) \\
&\quad dV_{g_{T-\tau_{\ell+1}}}(y_{\ell+1}) \cdots dV_{g_{T-\tau_k}}(y_k) \\
&=: f_\tau(X_{\tau_1}, ..., X_{\tau_\ell}, X_\tau). \qquad (3.5)
\end{aligned}$$

Note that, for $(x_1, \dots, x_\ell)$ fixed, $(x, \tau) \to f_\tau(x_1, \dots, x_\ell, x)$ is uniformly Lipschitz in τ and solves $(\partial_\tau + \Delta^{(\ell+1)})f_\tau = 0$, where $\Delta^{(\ell+1)}$ acts on the last entry.

Consider the lift $\tilde{f}_\tau := f_\tau \circ \otimes_1^{\ell+1}\pi_1 \circ \otimes_1^{\ell+1}\pi$. Also let $\tilde{F}_\tau := F_\tau \circ \Pi$, where $\Pi : P\mathcal{F} \to P\mathcal{M}$. Then we have that $\tilde{F}_\tau(U) = \tilde{f}_\tau(U_{\tau_1}, ..., U_{\tau_\ell}, U_\tau)$, which satisfies $(D_\tau + \Delta_H^{(\ell+1)})\tilde{f}_\tau = 0$ by applying Proposition 2.5. Also note that herein we shall denote the vector $(H_1\tilde{f}, ...H_n\tilde{f})$ by $H\tilde{f}$.

Then, by Proposition 2.9, we calculate

$$\begin{aligned}
d\tilde{F}_\tau(U) &= d(\tilde{f}_\tau(U_{\tau_1}, ..., U_{\tau_\ell}, U_\tau)) \\
&= \langle H^{(\ell+1)}(\tilde{f})(U_{\tau_1}, ..., U_{\tau_\ell}, U_\tau), dW_\tau \rangle \\
&\quad + \left(D_\tau + \Delta_H^{(\ell+1)}\right) \tilde{f}_\tau(U_{\tau_1}, ..., U_{\tau_\ell}, U_\tau)\, d\tau \\
&= \langle H^{(\ell+1)}(\tilde{f})(U_{\tau_1}, ..., U_{\tau_\ell}, U_\tau), dW_\tau \rangle. \qquad (3.6)
\end{aligned}$$

Next, we project down to M by Proposition 2.5 as follows

$$\begin{aligned}
H_a^{(\ell+1)}\tilde{f}_\tau(U_{\tau_1}, ..., U_{\tau_\ell}, U_\tau) &= (U_\tau e_a)^* \tilde{f}_\tau(U_{\tau_1}, ..., U_{\tau_\ell}, U_\tau) \\
&= (U_\tau e_a) f_\tau(X_{\tau_1}, ...X_{\tau_\ell}, X_\tau) \\
&= \langle U_\tau e_a, \mathrm{grad}_{g_{T-\tau}}^{(\ell+1)} f_\tau(X_{\tau_1}, ..., X_{\tau_\ell}, X_\tau) \rangle_{(T_{X_\tau}M, g_{T-\tau})} \\
&= \langle P_\tau U_\tau e_a, P_\tau \mathrm{grad}_{g_{T-\tau}}^{(\ell+1)} f_\tau(X_{\tau_1}, ..., X_{\tau_\ell}, X_\tau) \rangle_{(T_x M, g_T)} \\
&= \langle U_0 e_a, \nabla_\tau^{||} F_\tau(\gamma) \rangle_{(T_x M, g_T)}, \qquad (3.7)
\end{aligned}$$

whence

$$H_a^{(\ell+1)}(\tilde{f}_\tau)\, dW_\tau^a = \langle \nabla_\tau^{||} F_\tau(\gamma), U_0 e_a\rangle\, dW_\tau^a = \langle \nabla_\tau^{||} F_\tau(\gamma), dW_\tau\rangle, \tag{3.8}$$

and we deduce that

$$dF_\tau(\gamma) = d\tilde{F}_\tau(U) = \langle \nabla_\tau^{||} F_\tau(\gamma), dW_\tau\rangle \tag{3.9}$$

to complete the proof. □

Theorem 3.3. *(Evolution of the parallel gradient) If $F_\tau : P_{(x,T)}\mathcal{M} \to \mathbb{R}$ is a martingale on parabolic path space, and $\sigma \geqslant 0$ is fixed, then $\nabla_\sigma^{||} F_\tau : P_{(x,T)}\mathcal{M} \to (T_{(x,T)}M, g_T)$ satisfies the stochastic differential equation*

$$d(\nabla_\sigma^{||} F_\tau) = \langle \nabla_\tau^{||}\nabla_\sigma^{||} F_\tau, dW_\tau\rangle + \frac{1}{2}(\dot{g} + 2\mathrm{Rc})_\tau(\nabla_\tau^{||} F_\tau)\, d\tau + \nabla_\sigma^{||} F_\sigma d\delta_\sigma(\tau), \tag{3.10}$$

where $\langle (\dot{g}+2\mathrm{Rc})_\tau(v), w\rangle_{(T_xM, g_T)} = (\dot{g}_t + 2\mathrm{Rc}_{g_t})|_{t=T-\tau}(P_\tau^{-1}v, P_\tau^{-1}w)$ and $\dot{g} = \frac{d}{dt}g$.

Proof. As F_τ is Σ_τ-measurable, we have that $\nabla_\sigma^{||} F_\tau \equiv 0$ for $\sigma > \tau$. Noting $d(\nabla_\sigma^{||} F_\tau)$ is continuous except for a jump discontinuity at $\sigma = \tau$, we calculate

$$\begin{aligned} d(\nabla_\sigma^{||} F_\tau) &= d(\nabla_\sigma^{||} F_\tau)_{\mathrm{cont}} + \left(\nabla_\sigma^{||} F_{\sigma^+} - \nabla_\sigma^{||} F_{\sigma^-}\right) d\delta_\sigma(\tau) \\ &= d(\nabla_\sigma^{||} F_\tau)_{\mathrm{cont}} + \nabla_\sigma^{||} F_\sigma\, d\delta_\sigma(\tau). \end{aligned} \tag{3.11}$$

It remains to show that the identity holds for $\sigma \leqslant \tau$. In particular, we'll show that the continuous parts of the measures agree.

By approximation (cf. [HN18a, Sec 2.4]), it suffices to prove the theorem in the case where F_τ is a martingale induced by a k-point cylinder function as in the previous proof. Now, as σ is fixed, it is sufficient for us to consider the evolution equation for $\tau \in (\tau_\ell, \tau_{\ell+1})$, using the parallel gradient from example 2.20,

$$\nabla_\sigma^{||} F_\tau(\gamma) = \sum_{\tau_j \geqslant \sigma} P_{\tau_j} \nabla^{(j)} f_\tau(X_{\tau_1}, ..., X_{\tau_\ell}, X_\tau) + P_\tau \nabla^{(\ell+1)} f_\tau(X_{\tau_1}, ..., X_{\tau_\ell}, X_\tau), \tag{3.12}$$

which can be lifted to the frame bundle and represented by Proposition 2.5 as

$$
\begin{aligned}
G_a(U) :&= \langle U_0 e_a, \nabla^{||}_\sigma F_\tau(\Pi U)\rangle \\
&= \sum_{\tau_j \geqslant \sigma} \langle U_{\tau_j} e_a, \nabla^{(j)} f_\tau(X_{\tau_1}, ..., X_{\tau_\ell}, X_\tau)\rangle \\
&\quad + \langle U_\tau e_a, \nabla^{(\ell+1)} f_\tau(X_{\tau_1}, ..., X_{\tau_\ell}, X_\tau)\rangle \\
&= \sum_{\tau_j \geqslant \sigma} H_a^{(j)} \tilde{f}_\tau(U_{\tau_1}, ..., U_{\tau_\ell}, U_\tau) + H_a^{(\ell+1)} \tilde{f}_\tau(U_{\tau_1}, ..., U_{\tau_\ell}, U_\tau). \quad (3.13)
\end{aligned}
$$

Applying Propositions 2.8 and 2.9 and the fact that $(D_\tau + \Delta_H^{(\ell+1)})\tilde{f}_\tau = 0$ from the previous proof, we have that

$$
\begin{aligned}
&dG_a(U) \\
&= \sum_{\tau_k \geqslant \sigma} H_b^{(\ell+1)} H_a^{(k)} \tilde{f}_\tau(U_{\tau_1}, ..., U_{\tau_\ell}, U_\tau)\, dW_\tau^b \\
&\quad + H_b^{(\ell+1)} H_a^{(\ell+1)} \tilde{f}_\tau(U_{\tau_1}, ..., U_{\tau_\ell}, U_\tau)\, dW_\tau^b \\
&\quad + \sum_{\tau_k \geqslant \sigma} \left(D_\tau + \Delta_H^{(\ell+1)}\right) H_a^{(k)} \tilde{f}_\tau(U_{\tau_1}, ..., U_{\tau_\ell}, U_\tau)\, d\tau \\
&\quad + \left(D_\tau + \Delta_H^{(\ell+1)}\right) H_a^{(\ell+1)} \tilde{f}_\tau(U_{\tau_1}, ..., U_{\tau_\ell}, U_\tau)\, d\tau \\
&= \sum_{\tau_k \geqslant \sigma} H_b^{(\ell+1)} H_a^{(k)} \tilde{f}_\tau(U_{\tau_1}, ..., U_{\tau_\ell}, U_\tau)\, dW_\tau^b \\
&\quad + H_b^{(\ell+1)} H_a^{(\ell+1)} \tilde{f}_\tau(U_{\tau_1}, ..., U_{\tau_\ell}, U_\tau)\, dW_\tau^b \\
&\quad + \sum_{\tau_k \geqslant \sigma} \left(H_a^{(k)} \left(D_\tau + \Delta_H^{(\ell+1)}\right) + [D_\tau + \Delta_H^{(\ell+1)}, H_a^{(k)}]\right) \tilde{f}_\tau(U_{\tau_1}, ..., U_{\tau_\ell}, U_\tau)\, d\tau \\
&+ \left(H_a^{(\ell+1)} \left(D_\tau + \Delta_H^{(\ell+1)}\right) + [D_\tau + \Delta_H^{(\ell+1)}, H_a^{(\ell+1)}]\right) \tilde{f}_\tau(U_{\tau_1}, ..., U_{\tau_\ell}, U_\tau)\, d\tau \\
&= \sum_{\tau_k \geqslant \sigma} H_b^{(\ell+1)} H_a^{(k)} \tilde{f}_\tau(U_{\tau_1}, ..., U_{\tau_\ell}, U_\tau)\, dW_\tau^b \qquad (3.14) \\
&\quad + H_b^{(\ell+1)} H_a^{(\ell+1)} \tilde{f}_\tau(U_{\tau_1}, ..., U_{\tau_\ell}, U_\tau)\, dW_\tau^b \\
&\quad + \frac{1}{2}(\dot{\tilde{g}} + 2\widetilde{\mathrm{Rc}})_{ab}(U_\tau) H_b^{(\ell+1)} \tilde{f}_\tau(U_{\tau_1}, ..., U_{\tau_\ell}, U_\tau)\, d\tau.
\end{aligned}
$$

Finally, we project down onto $\mathcal{M}$ by Proposition 2.5 as follows,

$$\begin{aligned}
&H_b^{(\ell+1)} H_a^{(\ell+1)} \tilde{f}_\tau(U_{\tau_1}, ..., U_{\tau_\ell}, U_\tau) \\
&\quad = (U_\tau e_b)^* (U_\tau e_a)^* \tilde{f}_\tau(U_{\tau_1}, ..., U_{\tau_\ell}, U_\tau) \\
&\quad = \left(\nabla^{(\ell+1)} \nabla^{(\ell+1)} f_\tau(X_{\tau_1}, ..., X_{\tau_\ell}, X_\tau)\right) (U_\tau e_b, U_\tau e_a) \\
&\quad = \langle U_\tau e_b \otimes U_\tau e_a, \nabla^{(\ell+1)} \nabla^{(\ell+1)} f_\tau(X_{\tau_1}, ..., X_{\tau_\ell}, X_\tau) \rangle \\
&\quad = \left\langle U_0 e_b \otimes U_0 e_a, (P_\tau \otimes P_\tau) \left(\nabla^{(\ell+1)} \nabla^{(\ell+1)} f_\tau(X_{\tau_1}, ..., X_{\tau_\ell}, X_\tau)\right)\right\rangle, \qquad (3.15)
\end{aligned}$$

and similarly,

$$\begin{aligned}
&H_b^{(\ell+1)} H_a^{(k)} \tilde{f}_\tau(U_{\tau_1}, ..., U_{\tau_\ell}, U_\tau) \\
&\quad = \langle U_\tau e_b \otimes U_{\tau_k} e_a, \nabla^{(\ell+1)} \nabla^{(k)} f_\tau(X_{\tau_1}, ..., X_{\tau_\ell}, X_\tau) \rangle \\
&\quad = \left\langle U_0 e_b \otimes U_0 e_a, (P_\tau \otimes P_{\tau_k}) \left(\nabla^{(\ell+1)} \nabla^{(k)} f_\tau(X_{\tau_1}, ..., X_{\tau_\ell}, X_\tau)\right)\right\rangle, \qquad (3.16)
\end{aligned}$$

whence

$$\begin{aligned}
&\sum_{\tau_k \geqslant \sigma} (H_b^{(\ell+1)} H_a^{(k)}) (\tilde{f}_\tau)\, dW_\tau^b + (H_b^{(\ell+1)} H_a^{(\ell+1)}) (\tilde{f}_\tau)\, dW_\tau^b \\
&\quad = \left\langle \sum_{\tau_k \geqslant \sigma} (P_\tau \otimes P_{\tau_k}) \nabla^{(\ell+1)} \nabla^{(k)} f_\tau, U_0 e_b \otimes U_0 e_a \right\rangle dW_\tau^b \\
&\qquad + \langle (P_\tau \otimes P_\tau) \nabla^{(\ell+1)} \nabla^{(\ell+1)} f_\tau, U_0 e_b \otimes U_0 e_a \rangle\, dW_\tau^b \\
&\quad = \left\langle \sum_{\tau_k \geqslant \sigma} (P_\tau \otimes P_{\tau_k}) \nabla^{(\ell+1)} \nabla^{(k)} f_\tau, dW_\tau \otimes U_0 e_a \right\rangle \\
&\qquad + \langle (P_\tau \otimes P_\tau) \nabla^{(\ell+1)} \nabla^{(\ell+1)} f_\tau, dW_\tau \otimes U_0 e_a \rangle \\
&\quad = \left\langle \sum_{\tau_k \geqslant \sigma} (P_\tau \otimes P_{\tau_k}) \nabla^{(\ell+1)} \nabla^{(k)} f_\tau + (P_\tau \otimes P_\tau) \nabla^{(\ell+1)} \nabla^{(\ell+1)} f_\tau, dW_\tau \otimes U_0 e_a \right\rangle \\
&\quad = \langle \nabla_\tau^{||} \nabla_\sigma^{||} F_\tau(\gamma), dW_\tau \otimes U_0 e_a \rangle. \qquad (3.17)
\end{aligned}$$

Finally, we check that

$$\begin{aligned}
&(\tilde{\dot{g}} + 2\widetilde{\mathrm{Rc}})_{ab}(U_\tau) H_b^{(\ell+1)} \tilde{f}_\tau(U_{\tau_1}, ..., U_{\tau_\ell}, U_\tau)\, d\tau \\
&\quad = (\dot{g} + 2\mathrm{Rc})_{\pi(U_\tau)}(U_\tau e_a, U_\tau e_b)\langle \nabla^{(\ell+1)} f_\tau(X_{\tau_1}, \ldots, X_{\tau_\ell}, X_\tau), U_\tau e_b\rangle\, d\tau \\
&\quad = (\dot{g} + 2\mathrm{Rc})_{\pi(U_\tau)} \Big(U_\tau e_a, \langle \nabla^{(\ell+1)} f_\tau(X_{\tau_1}, \ldots, X_{\tau_\ell}, X_\tau), U_\tau e_b\rangle U_\tau e_b \Big)\, d\tau \\
&\quad = (\dot{g} + 2\mathrm{Rc})_{\pi(U_\tau)} \Big(\nabla^{(\ell+1)} f_\tau(X_{\tau_1}, \ldots, X_{\tau_\ell}, X_\tau), U_\tau e_a \Big)\, d\tau \\
&\quad = (\dot{g} + 2\mathrm{Rc})|_{t=T-\tau}(P_\tau^{-1} \nabla_\tau^{||} F_\tau, P_\tau^{-1} U_0 e_a)\, d\tau \\
&\quad = \langle (\dot{g} + 2\mathrm{Rc})_\tau (\nabla_\tau^{||} F_\tau)\, d\tau, U_0 e_a \rangle
\end{aligned} \tag{3.18}$$

which completes the proof. □

We now turn to the proof of Theorem 1.1.

Proof of Theorem 1.1. As F_τ is Σ_τ-measurable, we have that $\nabla_\sigma^{||} F_\tau \equiv 0$ for $\sigma > \tau$. Noting $d(\nabla_\sigma^{||} F_\tau)$ is continuous except for the jump discontinuity at $\sigma = \tau$, we calculate

$$\begin{aligned}
d|\nabla_\sigma^{||} F_\tau|^2 &= 2\langle \nabla_\sigma^{||} F_\tau, d(\nabla_\sigma^{||} F_\tau)\rangle \\
&= 2\Big\langle \nabla_\sigma^{||} F_\tau, d(\nabla_\sigma^{||} F_\tau)\rangle_{\mathrm{cont}} + \Big(\nabla_\sigma^{||} F_{\sigma^+} - \nabla_\sigma^{||} F_{\sigma^-} \Big)\, d\delta_\sigma(\tau) \Big\rangle \\
&= d(|\nabla_\sigma^{||} F_\tau|^2)_{\mathrm{cont}} + 2|\nabla_\sigma^{||} F_\sigma|^2\, d\delta_\sigma(\tau).
\end{aligned} \tag{3.19}$$

It remains to show that the identity holds for $\sigma \leqslant \tau$. In particular, it remains to show that the continuous parts of the measures agree.

In the rightly-continuous case by Itô calculus and Theorem 3.3, we calculate the quadratic variation $d[\nabla_\sigma^{||} F_\tau, \nabla_\sigma^{||} F_\tau] = 2|\nabla_\tau^{||} \nabla_\sigma^{||} F_\tau|^2\, d\tau$ for $\sigma \leqslant \tau$ and then

$$\begin{aligned}
d(|\nabla_\sigma^{||} F_\tau|^2) &= 2\langle \nabla_\sigma^{||} F_\tau, d(\nabla_\sigma^{||} F_\tau)\rangle + d[\nabla_\sigma^{||} F_\tau, \nabla_\sigma^{||} F_\tau] \\
&= \langle \nabla_\tau^{||} |\nabla_\sigma^{||} F_\tau|^2, dW_\tau\rangle + (\dot{g} + 2\mathrm{Rc})_\tau (\nabla_\tau^{||} F_\tau, \nabla_\sigma^{||} F_\tau)\, d\tau \\
&\quad + 2|\nabla_\tau^{||} \nabla_\sigma^{||} F_\tau|^2\, d\tau,
\end{aligned} \tag{3.20}$$

which concludes the proof. □

Corollary 3.4. *(Bochner) The generalized Bochner formula on $P\mathcal{M}$ (Theorem 1.1) reduces to the standard Bochner formula (Lemma 3.1) in the case of* 1*-point functions. That is, the evolution of $|\nabla H_{T-\tau_1,T-\tau} f|^2$ for $\tau \leqslant \tau_1$ is given by*

$$\begin{aligned} &\frac{1}{2}\left(\partial_\tau + \Delta_{g_{T-\tau}}\right)|\nabla H_{T-\tau_1,T-\tau} f|^2 \\ &\quad = |\nabla^2 H_{T-\tau_1,T-\tau} f|^2 + \frac{1}{2}(\dot{g} + 2\mathrm{Rc})|_{t=T-\tau}(\nabla H_{T-\tau_1,T-\tau} f, \nabla H_{T-\tau_1,T-\tau} f). \end{aligned} \tag{3.21}$$

Proof. Fix $\sigma = 0$ in the evolution equation from Theorem 1.1. Next, we shall compute the evolution of $|\nabla_0^{||} F_\tau|^2$, where

$$F_\tau(\gamma) := \begin{cases} H_{T-\tau_1,T-\tau} f(\pi_1\gamma_\tau), \ \tau < \tau_1 \\ f(\pi_1\gamma_{\tau_1}), \ \tau \geqslant \tau_1 \end{cases} \tag{3.22}$$

is the martingale induced by $f(\pi_1\gamma_{\tau_1})$. Then, for $\tau \in [0, \tau_1]$, we calculate

$$|\nabla_0^{||} F_\tau|(\gamma) = |\nabla_\tau^{||} F_\tau|(\gamma) = |\nabla H_{T-\tau_1,T-\tau} f|(\pi_1\gamma_\tau) \tag{3.23}$$

as well as

$$|\nabla_0^{||}\nabla_\tau^{||} F_\tau|(\gamma) = |\nabla^2 H_{T-\tau_1,T-\tau} f|(\pi_1\gamma_\tau). \tag{3.24}$$

By Theorem 1.1, we then deduce that

$$\begin{aligned} &d(|\nabla H_{T-\tau_1,T-\tau} f|^2) - \langle \nabla_\tau^{||}|\nabla H_{T-\tau_1,T-\tau}|^2, dW_\tau\rangle \\ &\quad = 2|\nabla^2 H_{T-\tau_1,T-\tau} f|^2\, d\tau + (\dot{g} + 2\mathrm{Rc})|_{t=T-\tau}(\nabla H_{T-\tau_1,T-\tau} f, \nabla H_{T-\tau_1,T-\tau} f)\, d\tau \end{aligned} \tag{3.25}$$

Moreover, for process $X_\tau = |\nabla H_{T-\tau_1,T-\tau} f|^2(\pi_1\gamma_\tau)$, by applying Itô calculus as in Proposition 2.9, we have that

$$\begin{aligned} &d(|\nabla H_{T-\tau_1,T-\tau} f|^2) - \langle \nabla_\tau^{||}|\nabla H_{T-\tau_1,T-\tau}|^2, dW_\tau\rangle \\ &\quad = \left(\partial_\tau + \Delta_{g_{T-\tau}}\right)|\nabla H_{T-\tau_1,T-\tau} f|^2\, d\tau. \end{aligned} \tag{3.26}$$

Therefore, by comparing equations (3.25) and (3.26), we conclude that

$$\begin{aligned} &\frac{1}{2}\left(\partial_\tau + \Delta_{g_{T-\tau}}\right)|\nabla H_{T-\tau_1,T-\tau} f|^2 \\ &\quad = |\nabla^2 H_{T-\tau_1,T-\tau} f|^2 + \frac{1}{2}(\dot{g} + 2\mathrm{Rc})|_{t=T-\tau}(\nabla H_{T-\tau_1,T-\tau} f, \nabla H_{T-\tau_1,T-\tau} f), \end{aligned} \tag{3.27}$$

which completes the proof. □

4

APPLICATIONS OF THE BOCHNER FORMULA ON PARABOLIC PATH SPACE

We shall now proceed by applying the Bochner formula on path space to both characterize the Ricci flow and develop gradient and Hessian estimates for martingales on parabolic path space.

4.1 PROOF OF THE BOCHNER INEQUALITY ON PARABOLIC PATH SPACE

Proof of Theorem 1.2. Using the formalism developed in the last section, we shall prove the equivalencies between the main estimates that characterize the Ricci flow.

(R1) $\implies$ (C1) $\implies$ (C2) $\implies$ (C3): If $(M, g_t)_{t \in I}$ evolves by Ricci flow $\partial_t g_t = -2\mathrm{Rc}_{g_t}$ and $F_\tau : P_{(x,T)}\mathcal{M} \to \mathbb{R}$ is a martingale on parabolic path space, then Theorem 1.1 gives

$$d|\nabla^{||}_\sigma F_\tau|^2 = \langle \nabla^{||}_\tau |\nabla^{||}_\sigma F_\tau|^2, dW_\tau \rangle + 2|\nabla^{||}_\tau \nabla^{||}_\sigma F_\tau|^2 \, d\tau + 2|\nabla^{||}_\sigma F_\sigma|^2 d\delta_\sigma(\tau), \quad (4.1)$$

thus proving (C1).

Next, to show (C2), calculate

$$|\Delta^{||}_{\sigma,\tau} F_\tau|^2 = \left| g^{ij} \left(\nabla^{||}_\sigma \nabla^{||}_\tau F_\tau \right)_{ij} \right|^2 \leqslant |g^{ij}|^2 |\nabla^{||}_\sigma \nabla^{||}_\tau F_\tau|^2 = n |\nabla^{||}_\sigma \nabla^{||}_\tau F_\tau|^2, \quad (4.2)$$

and finally show (C3) by simply dropping the non-negative term $\frac{2}{n}|\Delta^{||}_{\sigma,\tau} F_\tau|^2$ in (C2).

(C1) $\implies$ (C4) $\iff$ (C5): To prove (C4), first apply Itô's lemma to the left-hand side of the full Bochner inequality (C1) to get

$$\begin{aligned}
&2|\nabla^{||}_{\sigma}F_{\tau}|\langle\nabla^{||}_{\tau}|\nabla^{||}_{\sigma}F_{\tau}|, dW_{\tau}\rangle + 2|\nabla^{||}_{\tau}\nabla^{||}_{\sigma}F_{\tau}|^2\,d\tau + 2|\nabla^{||}_{\sigma}F_{\sigma}|^2 d\delta_{\sigma}(\tau)\\
&\quad = \langle\nabla^{||}_{\tau}|\nabla^{||}_{\sigma}F_{\tau}|^2, dW_{\tau}\rangle + 2|\nabla^{||}_{\tau}\nabla^{||}_{\sigma}F_{\tau}|^2\,d\tau + 2|\nabla^{||}_{\sigma}F_{\sigma}|^2\delta_{\sigma}(\tau)\\
&\quad \overset{(C1)}{\leqslant} d|\nabla^{||}_{\sigma}F_{\tau}|^2\\
&\quad = 2|\nabla^{||}_{\sigma}F_{\tau}|\,d|\nabla^{||}_{\sigma}F_{\tau}| + d\left[|\nabla^{||}_{\sigma}F_{\tau}|, |\nabla^{||}_{\sigma}F_{\tau}|\right]_{\tau}\\
&\quad = 2|\nabla^{||}_{\sigma}F_{\tau}|\,d|\nabla^{||}_{\sigma}F_{\tau}| + 2|\nabla^{||}_{\tau}|\nabla^{||}_{\sigma}F_{\tau}||^2\,d\tau\\
&\quad \leqslant 2|\nabla^{||}_{\sigma}F_{\tau}|\,d|\nabla^{||}_{\sigma}F_{\tau}| + 2|\nabla^{||}_{\tau}\nabla^{||}_{\sigma}F_{\tau}|^2\,d\tau. \qquad (4.3)
\end{aligned}$$

Rearranging this inequality and applying (C1), we derive (C4), namely

$$d|\nabla^{||}_{\sigma}F_{\tau}| \geqslant \langle\nabla_{\tau}|\nabla^{||}_{\sigma}F_{\tau}|, dW_{\tau}\rangle + |\nabla^{||}_{\sigma}F_{\sigma}|d\delta_{\sigma}(\tau). \qquad (4.4)$$

Finally, (C4) is satisfied if and only if F_{τ} is a submartingale (cf. Theorem 3.2) (C5) also holds. The remaining equivalencies will be proved in tandem with the results in the subsequent few theorems. □

4.2 PROOF OF GRADIENT ESTIMATES FOR MARTINGALES

Proof of Theorem 1.3. (C5) $\implies$ (G1) $\implies$ (G2): The implication of (G1) follows from the definition that if $\tau \to |\nabla^{||}_{\sigma}F_{\tau}|$ is a submartingale for every $\sigma \geqslant 0$, then for $\tilde{\tau} \geqslant \tau$, $|\nabla^{||}_{\sigma}F_{\tau}| \leqslant \mathbb{E}\left[|\nabla^{||}_{\sigma}F_{\tilde{\tau}}||\Sigma_{\tau}\right]$. Finally, to prove (G2), apply (G1) and Cauchy-Schwarz to get

$$|\nabla^{||}_{\sigma}F_{\tau}|^2 \leqslant \left(\mathbb{E}\left[|\nabla^{||}_{\sigma}F_{\tilde{\tau}}||\Sigma_{\tau}\right]\right)^2 \leqslant \mathbb{E}\left[|\nabla^{||}_{\sigma}F_{\tilde{\tau}}|^2|\Sigma_{\tau}\right]\cdot\mathbb{E}[1|\Sigma_{\tau}] = \mathbb{E}\left[|\nabla^{||}_{\sigma}F_{\tilde{\tau}}|^2|\Sigma_{\tau}\right]. \qquad (4.5)$$

The converse implications shall be proven along with later results. □

4.3 PROOF OF HESSIAN ESTIMATES FOR MARTINGALES

Proof of Theorem 1.4. (C1) $\implies$ (H1): To prove (H1), fix $\sigma \geqslant 0$ and then integrate (C1) from 0 to T as well as take expectations

$$\begin{aligned}
&\mathbb{E}_{(x,T)}\left[|\nabla_\sigma^{||}F|^2\right] - \mathbb{E}_{(x,T)}\left[|\nabla_\sigma^{||}F_\sigma|^2|\right] \\
&\overset{(C1)}{\geqslant} \mathbb{E}_{(x,T)}\left[\int_0^T \langle \nabla_\tau |\nabla_\sigma^{||}F_\tau|^2, dW_\tau\rangle\right] + 2\mathbb{E}_{(x,T)}\left[\int_0^T |\nabla_\tau^{||}\nabla_\sigma^{||}F_\tau|^2\, d\tau\right] \\
&= 2\mathbb{E}_{(x,T)}\left[\int_0^T |\nabla_\tau^{||}\nabla_\sigma^{||}F_\tau|^2\, d\tau\right].
\end{aligned} \tag{4.6}$$

(H1) $\implies$ (H2): To prove (H2), apply Itô isometry and then integrate (H1) from 0 to T with respect to σ as well as take expectations

$$\begin{aligned}
\mathbb{E}_{(x,T)}\left[(F - \mathbb{E}_{(x,T)}[F])^2\right] &= \mathbb{E}_{(x,T)}\left[\int_0^T |\nabla_\sigma^{||}F_\sigma|^2\, d\sigma\right] \\
&\overset{(H1)}{\leqslant} \mathbb{E}_{(x,T)}\left[\int_0^T |\nabla_\sigma^{||}F|^2\, d\sigma\right] \\
&\quad - 2\mathbb{E}_{(x,T)}\left[\int_0^T \int_0^T |\nabla_\tau^{||}\nabla_\sigma^{||}F_\tau|^2\, d\tau\, d\sigma\right].
\end{aligned} \tag{4.7}$$

(R1) $\implies$ (H3): To prove (H3), let $G = F^2$ and consider the evolution equation for $X_\tau := G_\tau^{-1}|\nabla^{\mathcal{H}}G_\tau|^2 - 2G_\tau\log(G_\tau)$, which satisfies

$$\begin{aligned}
dX_\tau &= \langle \nabla_\tau^{||}X_\tau, dW_\tau\rangle + 2G_\tau\left(\int_0^T |\nabla_\tau^{||}\nabla_\sigma^{||}\log(G_\tau)|^2\, d\sigma\right) d\tau \\
&\quad + G_\tau^{-1}\left(\int_0^T (\dot{g} + 2\mathrm{Rc})_\tau(\nabla_\tau^{||}F_\tau, \nabla_\sigma^{||}F_\tau)\, d\sigma\right) d\tau \\
&\geqslant \langle \nabla_\tau^{||}X_\tau, dW_\tau\rangle + 2G_\tau\left(\int_0^T |\nabla_\tau^{||}\nabla_\sigma^{||}\log(G_\tau)|^2\, d\sigma\right) d\tau
\end{aligned} \tag{4.8}$$

by Itô calculus and Proposition 3.3 (cf. Proposition 4.23 of [HN18b]). Next, integrate the inequality (4.8) from 0 to T with respect to τ and take expectations to get

$$\mathbb{E}_{(x,T)}[X_T] - \mathbb{E}_{(x,T)}[X_0] \geqslant 2\mathbb{E}_{(x,T)}\left[G_\tau\left(\int_0^T |\nabla_\tau^{||}\nabla_\sigma^{||}\log(G_\tau)|^2\, d\sigma\right) d\tau\right], \tag{4.9}$$

and evaluating the two expectations in the difference, namely

$$\begin{aligned}\mathbb{E}_{(x,T)}[X_0] &= \mathbb{E}_{(x,T)}[G_0^{-1}|\nabla^{\mathcal{H}} G_0|^2 - 2G_0 \log(G_0)] \\ &= 0 - 2G_0 \log(G_0) \\ &= -2\mathbb{E}_{(x,T)}[F^2] \log\left(\mathbb{E}_{(x,T)}[F^2]\right) \end{aligned} \tag{4.10}$$

and

$$\begin{aligned}\mathbb{E}_{(x,T)}[X_T] &= \mathbb{E}_{(x,T)}\left[G^{-1}|\nabla^{\mathcal{H}} G|^2 - 2G\log(G)\right] \\ &= \mathbb{E}_{(x,T)}\left[F^{-2}|\nabla^{\mathcal{H}} F^2|^2\right] - 2\mathbb{E}_{(x,T)}\left[F^2\log(F^2)\right] \\ &= 4\mathbb{E}_{(x,T)}\left[|\nabla^{\mathcal{H}} F|^2\right] - 2\mathbb{E}_{(x,T)}\left[F^2\log(F^2)\right]. \end{aligned} \tag{4.11}$$

Finally we observe that

$$\mathbb{E}_{(x,T)}\left[|\nabla^{\mathcal{H}} F|^2\right] = \mathbb{E}_{(x,T)}\left[\int_0^T |\nabla_\sigma^{||} F|^2 \, d\sigma\right], \tag{4.12}$$

and then combine this and the aforementioned results to prove the claim. □

4.4 PROOF OF THE CHARACTERIZATIONS OF SOLUTIONS OF THE RICCI FLOW

The following result reproves a theorem by Haslhofer and Naber (cf. Theorem 1.22 of [HN18a]), characterizing solutions of the Ricci flow, using the Bochner formulas on path space that were developed in the previous section.

Proof of Theorem 1.5. (G1) $\implies$ (R2): To prove (R2), we evaluate (G1) at $\sigma = \tau = 0$,

$$\left|\nabla_x \mathbb{E}_{(x,t)}[F]\right| = |\nabla_x F_0| \overset{\text{(G1)}}{\leqslant} \mathbb{E}_{(x,T)}\left[|\nabla_0^{||} F| |\Sigma_0\right] = \mathbb{E}_{(x,T)}\left[|\nabla_0^{||} F|\right]. \tag{4.13}$$

(G2) $\implies$ (R3): To prove (R3), we evaluate (G2) at $\sigma, \tau = 0$ (and observe that $d[F,F]_\tau = 2|\nabla^{||}_\tau F_\tau|^2\, d\tau$ by Theorem 3.2),

$$\begin{aligned}\mathbb{E}_{(x,T)}\left[\frac{d[F,F]_\tau}{d\tau}\right] &= 2\mathbb{E}_{(x,T)}\left[|\nabla^{||}_\tau F_\tau|^2\right]\\ &\overset{\text{(G2)}}{\leqslant} 2\mathbb{E}_{(x,T)}\left[\mathbb{E}_{(x,T)}\left[|\nabla^{||}_\tau F|^2|\Sigma_\tau\right]\right]\\ &\leqslant 2\mathbb{E}_{(x,T)}\left[|\nabla^{||}_\tau F|^2\right].\end{aligned} \tag{4.14}$$

(G1) $\implies$ (R4): To prove (R4), we set $\sigma = \tau$ and take expectations

$$\mathbb{E}_{(x,T)}\left[|\nabla^{||}_\sigma F_\sigma|\right] \leqslant \mathbb{E}_{(x,T)}\left[\mathbb{E}_{(x,T)}\left[|\nabla^{||}_\sigma F||\Sigma_\sigma\right]\right] = \mathbb{E}_{(x,T)}\left[|\nabla^{||}_\sigma F|\right]. \tag{4.15}$$

Then follow the proof of (H3) in Theorem 1.4 and evaluate the expectation $\mathbb{E}_{(x,T)}[X_{\tau_j}]$ for $j \in \{1,2\}$, namely

$$\mathbb{E}_{(x,T)}[X_{\tau_j}] = \mathbb{E}_{(x,T)}\left[G_{\tau_j}^{-1}|\nabla^{\mathcal{H}} G_{\tau_j}|^2\right] - 2\mathbb{E}_{(x,T)}\left[G_{\tau_j}\log(G_{\tau_j})\right] \tag{4.16}$$

and taking differences, where $\mathbb{E}_{(x,T)}[X_{\tau_2}\,|\,\Sigma_{\tau_1}] - \mathbb{E}_{(x,T)}[X_{\tau_1}\,|\,\Sigma_{\tau_1}] \geqslant 0$, as in the earlier proof. It remains to check that

$$\begin{aligned}&\mathbb{E}_{(x,T)}\left[G_{\tau_2}^{-1}|\nabla^{\mathcal{H}} G_{\tau_2}|^2\,|\,\Sigma_{\tau_1}\right] - \mathbb{E}_{(x,T)}\left[G_{\tau_1}^{-1}|\nabla^{\mathcal{H}} G_{\tau_1}|^2\,|\,\Sigma_{\tau_1}\right]\\ &\quad = 4\mathbb{E}_{(x,T)}\left[|\nabla^{\mathcal{H}} F_{\tau_2}|^2\,|\,\Sigma_{\tau_1}\right]\\ &\quad = 4\mathbb{E}_{(x,T)}\left[\int_{\tau_1}^{\tau_2}|\nabla^{||}_\sigma F_{\tau_2}|^2\, d\sigma\right]\\ &\quad \leqslant 4\mathbb{E}_{(x,T)}\left[\int_{\tau_1}^{\tau_2}|\nabla^{||}_\sigma F|^2\, d\sigma\right] \qquad (\tau \to |\nabla^{||}_\sigma F_\tau|^2 \text{ is a submartingale})\\ &\quad = 4\mathbb{E}_{(x,T)}\left[\langle F, \mathcal{L}_{(\tau_1,\tau_2)}F\rangle_{\mathcal{H}}\right].\end{aligned} \tag{4.17}$$

(G2) $\implies$ (R5): To prove (R5), we set $\sigma = \tau$ and take expectations

$$\mathbb{E}_{(x,T)}\left[|\nabla^{||}_\sigma F_\sigma|^2\right] \leqslant \mathbb{E}_{(x,T)}\left[\mathbb{E}_{(x,T)}\left[|\nabla^{||}_\sigma F|^2|\Sigma_\sigma\right]\right] = \mathbb{E}_{(x,T)}\left[|\nabla^{||}_\sigma F|^2\right]. \tag{4.18}$$

Then follow the proof of (H2) in Theorem 1.4 and apply Itô isometry

$$\begin{aligned}\mathbb{E}_{(x,T)}\left[(F_{\tau_2}-F_{\tau_1})^2\,|\,\Sigma_{\tau_1}\right] &= \mathbb{E}_{(x,T)}\left[\int_0^T |\nabla^{||}_\sigma F_{\tau_2}|^2\,d\sigma\,\middle|\,\Sigma_{\tau_1}\right]\\ &= \mathbb{E}_{(x,T)}\left[\int_{\tau_1}^{\tau_2} |\nabla^{||}_\sigma F_{\tau_2}|^2\,d\sigma\right]\\ &\leqslant \mathbb{E}_{(x,T)}\left[\langle F, \mathcal{L}_{(\tau_1,\tau_2)}F\rangle_{\mathcal{H}}\right]. \end{aligned} \tag{4.19}$$

The converse implications shall be proven in the next section. □

4.5 CONVERSE IMPLICATIONS

We shall now prove the converse implications below.

Proof. (C3) $\Longrightarrow$ (R1): First fix $(x,T)\in\mathcal{M}$ and $v\in(T_xM,g_T)$ a unit vector and choose a smooth compactly supported $f_1: M\to\mathbb{R}$ such that

$$f_1(x)=0,\quad \nabla f_1(x)=v,\quad \nabla^2 f_1(x)=0 \tag{4.20}$$

using exponential coordinates. Consider the one-point cylinder function given by $F(\gamma)=f_1(\pi_M(\gamma(\varepsilon)))$, $F: P_{(x,T)}\mathcal{M}\to\mathbb{R}$ and observe for $\tau\leqslant\varepsilon$ that

$$\nabla^{||}_\tau F_\tau = P_\tau\nabla H_{T-\tau,T}f_1(\pi_M(\gamma(\tau))),\quad |\nabla^{||}_\tau\nabla^{||}_0 F_\tau| = |\nabla^2 H_{T-\tau,T}f_1|(\pi_M(\gamma(\tau))). \tag{4.21}$$

In particular, $\nabla^{||}_\tau F_\tau = v+o(\varepsilon)$ and $|\nabla^{||}_\tau\nabla^{||}_0 F_\tau|=o(\varepsilon)$. Then, by Theorem 1.1,

$$\tau\to|\nabla^{||}_0 F_\tau|^2-\int_0^\tau\left(2|\nabla^{||}_\rho\nabla^{||}_0F_\rho|^2+(\dot g+2\mathrm{Rc})_\rho(\nabla^{||}_\rho F_\rho,\nabla^{||}_0F_\rho)\right)d\rho \tag{4.22}$$

is a martingale. So, in particular,

$$|\nabla^{||}_0 F_0|^2=\mathbb{E}\left[|\nabla^{||}_0F_\varepsilon|^2\right]-\varepsilon(\dot g+2\mathrm{Rc})_\varepsilon(v,v)+o(\varepsilon). \tag{4.23}$$

Moreover, since $\tau\to|\nabla^{||}_0F_\tau|^2$ is a submartingale by (C3), it follows that

$$(\dot g+2\mathrm{Rc})_\varepsilon(v,v)\geqslant\varepsilon^{-1}o(\varepsilon). \tag{4.24}$$

Next choose a smooth compactly supported $f_2: M\times M\to\mathbb{R}$ such that

$$f_2(x,x)=0,\quad \nabla^{(1)}f_2(x,x)=2v,\quad \nabla^{(2)}f_2(x,x)=-v,\quad \nabla^2 f_2(x,x)=0, \tag{4.25}$$

for example $f_2(y,z) = 2f_1(y) - f_1(z)$. Consider the two-point cylinder function given by $F(\gamma) = f_2(\pi_M(\gamma(0)), \pi_M(\gamma(\varepsilon)))$, $F : P_{(x,T)}\mathcal{M} \to \mathbb{R}$ and observe for $\tau \leqslant \varepsilon$ that

$$\begin{cases} \nabla_0^{||} F_\tau & = \nabla^{(1)} f_2(x, \pi_M(\gamma(\tau))) + P_\tau \nabla H^{(2)}_{T-\tau,T} f_2(x, \pi_M(\gamma(\tau))) \\ \nabla_\tau^{||} F_\tau & = P_\tau \nabla H^{(2)}_{T-\tau,T} f_2(x, \pi_M(\gamma(\tau))) \\ |\nabla_\tau^{||} \nabla_0^{||} F_\tau| & \leqslant |\nabla^2 f_2|(x, \pi_M(\gamma(\tau))) + |\nabla^2 H^{(2)}_{T-\tau,T} f_2|(x, \pi_M(\gamma(\tau))). \end{cases} \tag{4.26}$$

In particular, $\nabla_0^{||} F_\tau = v + o(\varepsilon)$, $\nabla_\tau^{||} F_\tau = -v + o(\varepsilon)$ and $|\nabla_\tau^{||} \nabla_0^{||} F_\tau| = o(\varepsilon)$. Then, again by Theorem 1.1,

$$|\nabla_0^{||} F_0|^2 = \mathbb{E}\left[|\nabla_0^{||} F_\varepsilon|^2 \right] + \tau(\dot{g} + 2\mathrm{Rc})_\varepsilon(v,v) + o(\varepsilon). \tag{4.27}$$

Moreover, since $\tau \to |\nabla_0^{||} F_\tau|^2$ is a submartingale by (C3), it follows that

$$(\dot{g} + 2\mathrm{Rc})_\varepsilon(v,v) \leqslant \tau^{-1} o(\varepsilon). \tag{4.28}$$

We can then deduce that (R1) is satisfied by taking $\varepsilon \to 0^+$ in equations (4.24) and (4.28).

To check the remaining converse implications, one can substitute 1-point and 2-point cylinder functions as above. However, there are some alternative tools that can close the loop of equivalencies more readily. For example, applying the log-Sobolev equality to $F^2 = 1 + \varepsilon G$ in (R4) gives the Poincaré inequality in (R5). Moreover, dividing by $T - \tau$, taking $T - \tau \to 0^+$ and using the quadratic variation $d[F,F]_\tau = 2|\nabla_\tau^{||} F_\tau|^2\, d\tau$ (by Theorem 3.2), (R3) can be derived from (R5). In short, some implications can be done directly without the need to appeal to test functions each time. □

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