STAC63 Final Exam

Due by: Saturday, April 19 at 16:00 EST via Crowdmark
Number of questions: 6 **Time: 3 hours + 20 minutes** **Total points available: 70**

1. (8 points) Capa plays either one or two chess games every day, with the number of games that she plays on successive days being a Markov chain with transition probabilities

$$P_{1,1} = 0.2, \qquad P_{1,2} = 0.8, \qquad P_{2,1} = 0.4, \qquad P_{22} = 0.6.$$

Capa wins each game with probability $\frac{1}{2}$. Suppose she plays two games on Monday.

(a) (2 points) What is the probability that she wins all the games she plays on Tuesday?

(b) (3 points) What is the expected number of games that she plays on Wednesday?

(c) (3 points) In the long-run, on what proportion of days does Capa win all her games?

2. (15 points) Let $S = \mathbb{N} = \{0, 1, \dots\}$ be a state space and let $\{X_n\}$ be a Markov chain on S with $X_0 = a > 0$ and with transition probabilities given by $P_{00} = 1$, and for $i > 0$, $P_{i,i\pm 1} = \frac{1}{2}$, with $P_{ij} = 0$ otherwise. Also let $T := \inf\{j \geq 0 \,|\, X_j \in \{0, a+b\}\}$.

   (a) (3 points) Show that $\{X_n\}$ is a martingale. What is $\mathbb{E}[X_n]$?

   (b) (3 points) Compute $\mathbb{P}(X_T = a + b)$.

   (c) (3 points) Show that $X_n \xrightarrow{p} 0$.

   (d) (3 points) Show that $\{Y_n = X_n^2 - n\}$ is a martingale.

   (e) (3 points) Compute $\mathbb{E}[T]$.

3. (8 points) A certain scientific theory supposes that mistakes in cell division occur according to a Poisson process with rate 2.5 per year, and that an individual dies when 196 such mistakes have occurred. Let T denote the lifetime of an individual. Assuming this theory, find

   (a) (2 points) $\mathbb{E}[N(T)]$ and $\mathbb{V}(N(T))$,

   (b) (3 points) $\mathbb{E}[T]$,

   (c) (3 points) $\mathbb{V}(T)$.

4. (15 points) Let $\{N(t), t \geq 0\}$ be a non-homogeneous Poisson process with intensity function $\lambda(t)$. Also assume $s \leq t$.

   (a) (3 points) Compute $\mathbb{E}[N(s)N(t)]$.

   (b) (3 points) Compute $\mathbb{E}[N(s)N(t) \mid N(s)]$.

   (c) (3 points) Compute $\mathbb{E}[N(s)N(t) \mid N(t)]$.

   (d) (3 points) Find constants a_t and b_t such that

   $$\frac{N(t) - a_t}{b_t} \xrightarrow{d} \mathcal{N}(0, 1)$$

   as $t \to \infty$.

   (e) (3 points) Let $X(t) = N(t) - \int_0^t \lambda(s)\, ds$. Show that $\{X(t)\}_{t \geq 0}$ is a martingale.

5. (12 points) Let $\{B(t)\}_{t\geq 0}$ and $\{C(t)\}_{t\geq 0}$ be two independent Brownian motions. Also let $\rho \in (0,1)$ and $D(t) := \rho B(t) + \sqrt{1-\rho^2}C(t)$ for $t \geq 0$.

   (a) (3 points) Compute $\mathbb{E}[D(s)B(t)]$ for $0 \leq s \leq t$.

   (b) (3 points) Show that $\{D(t)\}_{t\geq 0}$ is a standard Brownian motion.

   (c) (3 points) Compute $\mathbb{E}[D(s)^2 B(t)^2]$ for $0 \leq s \leq t$.

   (d) (3 points) Let $X(t) = x_0 + \mu t + \sigma B(t)$. Compute $\mathbb{E}[X(t)^2]$.

6. (12 points) Let $Y(t) = e^{-\kappa t}\left(y_0 + \sigma \int_0^t e^{\kappa u}\, dB(u)\right)$. Also let $Z(t) = Y(t)^2$ for $t \geq 0$.

(a) (3 points) Compute $\mathbb{E}[Y(t)]$.

(b) (4 points) Compute $\mathrm{Cov}(Y(t), Y(s))$ for $0 \leq s \leq t$.

(c) (2 points) Is $Y(t)$ a Gaussian process? Is it a stationary process?

(d) (3 points) Using Itô's lemma, find a formula for $dZ(t)$.

(e) (2 points) *Bonus:* The answer is?