Midterm Test MATA31 (Winter 2020, UTSC)

- You have 2 hours for this midterm test.
- You are required to clearly fill in your name and student number below.
- No calculators, cheat sheets or other aids are permitted.
- There are 7 questions worth 10 points each.
- You only need 60 out of 70 points for full marks on this midterm.
- For full marks, you must write clear, consistent and complete solutions.

Problem 1

(10pts) Prove by induction on $n \in \mathbb{N}$ that

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}.$$

Continue your solution of Problem 1 here.

Problem 2

(10pts) Prove by contradiction that $\sqrt{3}$ is irrational.

Continue your solution of Problem 2 here.

Problem 3

(10pts) Find the domains of the following functions:

(a) $f(x) = \frac{1}{x-1} + \frac{1}{x-2}$

(b) $g(x) = \sqrt{1-x^2} + \sqrt{x^2-1}$

(c) $h(x) = \sqrt{1-x} + \sqrt{x-2}$

Problem 4

(10pts) Compute the following limits:

(a) $\lim_{x \to 0} \frac{1-\sqrt[3]{1-x^2}}{x^2}$

(b) $\lim_{x \to 1} (x^2-1)^3 \sin^3\left(\frac{1}{x-1}\right)$

(c) $\lim_{x \to \infty} \frac{x^2+\sin^3(x)}{(x+\sin(x))^2}$

Problem 5

(10pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $a, L \in \mathbb{R}$. Prove that, if

$$\lim_{x \to a} f(x) = L,$$

then

$$\lim_{x \to a/2} f(2x) = L.$$

Do a formal $\epsilon - \delta$ proof without using any limit laws.

Continue your solution of Problem 5 here.

Problem 6

(10pts) Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions. Let $h = f + g$ and $L, M \in \mathbb{R}$. Prove that, if

$$\lim_{x \to \infty} f(x) = L$$

and

$$\lim_{x \to \infty} g(x) = M,$$

then

$$\lim_{x \to \infty} h(x) = L + M.$$

Do a formal $\epsilon - \delta$ proof without using any limit laws.

Continue your solution of Problem 6 here.

Problem 7

(10pts) Let $L > 0$. A function $f : \mathbb{R} \to \mathbb{R}$ is L-Lipschitz if $\forall x, y \in \mathbb{R}$

$$|f(x) - f(y)| \leq L|x - y|.$$

Prove that a π-Lipschitz function is continuous. Do a formal $\epsilon - \delta$ proof.

Continue your solution of Problem 7 here.

Use this page if additional space is required. Clearly state the question number being answered and refer the marker to this page.