MAT235Y Assignment 4

Due by: Tuesday, August 25, 2020 at 10:00AM EDT via Crowdmark
Number of questions: 5 Total points available: 60

1. (12 points) In this question, we will compare the masses, m_B and m_E, and moments of inertia, I_B and I_E, of both a solid sphere $B = \{x^2 + y^2 + z^2 \leq 1\}$ and a solid truncated sphere (that is, sliced to remove a spherical cap) $E = \{x^2 + y^2 + z^2 \leq 1\} \cap \{x \leq \frac{1}{2}\}$ of uniform densities, ρ_B and ρ_E, that are precessing about the x-axis.

   (a) (5 points) Show that the masses of the unit and truncated spheres are $m_B = \frac{4\rho_B}{3}\pi$ and $m_E = \frac{9\rho_E}{8}\pi$.

   (b) (5 points) The moment of inertia I_V of an object V rotating about the x-axis is

   $$I_V = \iiint\limits_V (y^2 + z^2)\rho(x, y, z)\, dV.$$

   Using cylindrical coordinates, compute the moments of inertia I_B and I_E of the unit sphere and the truncated sphere.

   (c) (2 points) If unit sphere B and truncated sphere E have the same mass M, but different uniform densities, then which object will have the larger moment of inertia? Which will roll down an inclined plane fastest?

2. (12 points) Use Stokes' theorem to compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F} = (xyz - 2z)\,\mathbf{i} + xy^2\,\mathbf{j} + \sin(xyz)e^z\,\mathbf{k}$$

and C is the curve of intersection of the paraboloid $z = x^2 + y^2 + 1$ and the cylinder $x^2 + y^2 = 2$, oriented counterclockwise when viewed from above.

   (a) (2 points) Find the vector field curl $\mathbf{F}$ by direct computation (show your work!).

   (b) (4 points) Choose an oriented piecewise-smooth surface S, whose boundary is C. Provide a parametrization for your surface S.

   (c) (2 points) Using the parametrization of S, express a unit normal vector $\mathbf{n}$ that provides an orientation of S.

   (d) (4 points) Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.

3. (12 points) Use the Divergence theorem to compute the flux of the vector field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ across the positively oriented torus S obtained by rotating about the $z-$axis the circle in the $xz-$plane with center $(b, 0, 0)$ and radius $a < b$.

   (a) (1 point) Find div $\mathbf{F}$.

   (b) (1 point) Use the Divergence theorem to express the flux across S as a triple integral over some solid E. What is E?

   (c) (4 points) Parametrize E.

   Hint: use the parametrization of the torus from interesting problem 1 in Week 11 to parametrize E.

   (d) (4 points) Using (c), make a change of variables to reduce the triple integral to an iterated integral over a rectangular box.

   (e) (2 points) Compute the flux.

4. (12 points) Let $\vec{F} = \frac{1}{r^2}\,\hat{r}$ and let V be open, bounded and contain $(0,0,0)$.

   (a) (5 points) By showing that $\operatorname{div}(\vec{F}) = 0\,\forall \vec{x} \in \mathbb{R}^3 \setminus \{(0,0,0)\}$, show that

   $$\iint\limits_{\partial V} \vec{F} \cdot d\vec{S} = 4\pi.$$

   (b) (5 points) Letting S be the boundary of solid tetrahedron V with vertices at $(0,0,0)$, $(2,0,0)$, $(0,1,0)$ and $(0,0,4)$, compute the outward flux of the vector field

   $$\vec{G} = -\vec{\nabla}\left(\frac{q_1}{4\pi|\vec{r} - \vec{r}_1|}\right) - \vec{\nabla}\left(\frac{q_2}{4\pi|\vec{r} - \vec{r}_2|}\right) - \vec{\nabla}(e^x \sin(y) + z),$$

   where q_1, q_2 are constants, $\vec{r}_1 = (1/4, 1/4, 1/16)$ and $\vec{r}_2 = (2,2,2)$.

   (c) (2 points) Does your calculation in part (b) violate the divergence theorem? Explain.

5. (12 points) In diffraction theory, the *Fresnel-Kirchoff diffraction formula* gives an expression for the wave disturbance when a monochromatic spherical wave passes through an opening in an opaque screen. This exercise provides a preliminarily version of this formula. Let S a closed surface with positive orientation, $\mathbf{n}(x, y, z)$ the unit normal vector to S at the point (x, y, z) and V the volume of S. Let $E(x, y, z)$ and $U(x, y, z)$ be two smooth functions defined on $\mathbb{R}^3$.

   (a) (2 points) Prove the following product rule:
   $$\nabla \cdot (E\nabla U) = \nabla E \cdot \nabla U + E\nabla^2 U,$$
   where ∇^2 is the Laplacian (see week 11 problem list).

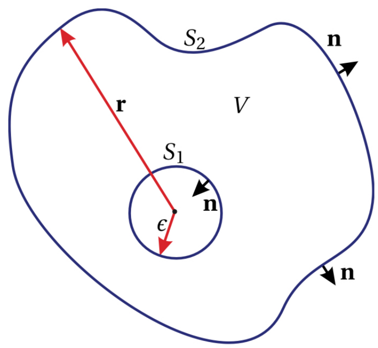
   (b) (2 points) We denote by $\frac{\partial U}{\partial n}$ the directional derivative of U along the direction of the vector $\mathbf{n}$. Use the divergence theorem with a suitable choice of the vector field $\mathbf{F}$ to show that
   $$\iint_S E\frac{\partial U}{\partial n}\, dS = \iiint_V \left(\nabla E \cdot \nabla U + E\nabla^2 U\right) dV$$

   (c) (2 points) Show that
   $$\iint_S \left(E\frac{\partial U}{\partial n} - U\frac{\partial E}{\partial n}\right) dS = \iiint_V \left(E\nabla^2 U - U\nabla^2 E\right) dV.$$

   (d) (1 points) We choose the following functions:
   $$U(x, y, z) = e^{lkr}/r$$
   $$E(x, y, z) = f(r),$$
   where $r = \sqrt{x^2 + y^2 + z^2} = |\mathbf{r}|$, f is a smooth function of r defined on $\mathbb{R}$ and l, k are constants such that E and U satisfy the *scalar Helmhotz equation*:
   $$\nabla^2 E + k^2 E = 0, \quad \nabla^2 U + k^2 U = 0.$$ [1]
   Consider a volume between a small sphere of radius ϵ at the origin and an outer surface of arbitrary shape. Hence the total surface that encloses the volume is comprised of two parts, that is $S = S_1 \cup S_2$. Show that
   $$\iint_{S_2} \left(E\frac{\partial U}{\partial n} - U\frac{\partial E}{\partial n}\right) dS = -\iint_{S_1} \left(E\frac{\partial U}{\partial n} - U\frac{\partial E}{\partial n}\right) dS.$$

   (e) (4 points) Show that
   $$\lim_{\epsilon\to 0} \iint_{S_1} \left(E\frac{\partial U}{\partial n} - U\frac{\partial E}{\partial n}\right) dS = 4\pi E(0, 0, 0).$$

[1] You don't need to verify this but the constant l needs to satisfy the equation $l^2 = -1$. Although it is possible in mathematics to wotk with this quantity as an imaginary number, your can assume that l is a regular constant.



Hint: Observe that $\nabla \boldsymbol{r} = -\boldsymbol{n}$ on S_1 and apply the chain rule to compute $\frac{\partial}{\partial n}\left(\frac{e^{lkr}}{r}\right)$ and $\frac{\partial E}{\partial n}$.

(f) (1 points) Express the value of $E(0,0,0)$ as a surface integral over S_2 and conclude that if we know E everywhere on the outer surface S_2, we can predict the value of E at the origin. When we apply this formula to a specific surface S_2 (i.e. a large spherical mask and an open aperture), we essentially obtain the *Fresnel-Kirchoff diffraction formula*.