MAT235Y Assignment 3

Due by: Saturday, August 1, 2020 at 10:00AM EDT via Crowdmark
Number of questions: 5 **Total points available: 60**

1. (15 points) The zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $s > 1$, plays an important role in many areas of mathematics, especially number theory (it can also be defined when s is a complex number). In 1736 Leonard Euler was able to prove that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

In this problem, your will prove this fact using what you know about double integrals and change of variables (the original proof used a different approach).

(a) (2 points) The double integral $\int_0^1 \int_0^1 \frac{1}{1-xy}\, dx\, dy$ exists as an improper integral and may be defined as the limit $\lim_{t\to 1^-} \int_0^t \int_0^t \frac{1}{1-xy}\, dx\, dy$. By expressing the integrand as a geometric series, show that

$$I = \int_0^1 \int_0^1 \frac{1}{1-xy}\, dx\, dy = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2).$$

(You may freely interchange the integral and sum.)

(b) (4 points) Find and sketch the region S in the uv-plane obtained by applying the following transformation T to the square $[0,1] \times [0,1]$:

$$x = \frac{u-v}{\sqrt{2}}, \quad y = \frac{u+v}{\sqrt{2}}.$$

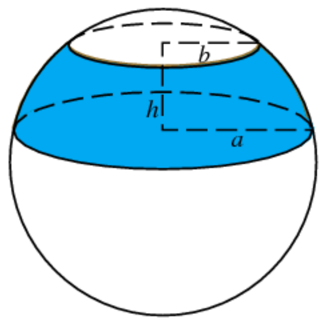
(c) (4 points) Using (a) and (b), express the integral I as a sum of two double integrals I_1 ans I_2. We denote by I_1 the integral over the part of S where $0 \le u \le \sqrt{2}/2$.

(d) (2 points) Using the identity $\sqrt{2 - (\sqrt{2}\,\sin\,\theta)^2} = \sqrt{2}\,\cos\,\theta$ when $\cos\,\theta \ge 0$, evaluate I_1. *Hint: find a suitable (one dimensional) change of variable.*

(e) (2 points) Using the identity $\arctan(\frac{\sqrt{2}-\sqrt{2}\sin\,\theta}{\sqrt{2}\,\cos\,\theta}) = \frac{1}{2}(\frac{1}{2}\,\pi - \theta)$, evaluate I_2. *Hint: find a suitable (one dimensional) change of variable.*

(f) (1 point) Conclude that $\zeta(2) = \frac{\pi^2}{6}$.

2. (8 points) A *zone* of a sphere is the portion of its surface which lies between two parallel planes which intersect the sphere. The *altitude* of the zone is the distance between the planes. Show that the surface area A of a zone depends only on its altitude h and the radius R of the sphere. [hint: you need to express A as a function of h and R.]



3. (9 points) In this question, we will find $\mathbb{E}(X_\sigma^n)$, the n^{th} moment of X_σ, where $X_\sigma \sim N(0, \sigma^2)$ is normally distributed with mean 0 and variance σ^2.

   (a) (3 points) A function $p_\sigma(x)$ is a probability measure on $\mathbb{R}$ if $p_\sigma(x) \geq 0 \, \forall x \in \mathbb{R}$ and $\int_{-\infty}^{\infty} p_\sigma(x)\, dx = 1$. By showing $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$, show that $p_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma}}$ is a probability measure.

   (b) (3 points) Completing the square, compute the moment generating function of X_σ

   $$M_{X_\sigma}(t) := \int_{-\infty}^{\infty} e^{tx} p_\sigma(x)\, dx = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{tx - \frac{x^2}{2\sigma}}\, dx.$$

   (c) (3 points) Using Taylor series, compute the n^{th} moment of X_σ

   $$\mathbb{E}(X_\sigma^n) := \int_{-\infty}^{\infty} x^n p_\sigma(x)\, dx = M_{X_\sigma}^{(n)}(0).$$

4. (8 points) Let $\vec{F} = (P, Q)$ be the vector field defined by

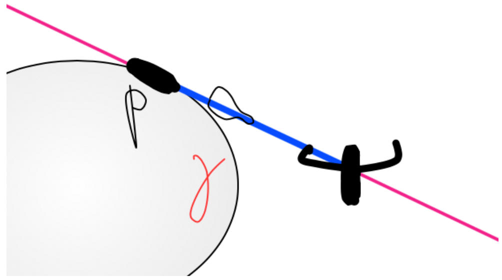
$$P(x,y) = \begin{cases} \frac{x+y}{x^2+y^2}, (x,y) \neq (0,0) \\ 0, (x,y) = (0,0) \end{cases} \qquad Q(x,y) = \begin{cases} \frac{-x+y}{x^2+y^2}, (x,y) \neq (0,0) \\ 0, (x,y) = (0,0). \end{cases}$$

(a) (3 points) Show that $\vec{F}$ is a gradient vector field in $\mathbb{R}^2 \setminus \{y = 0\}$.

(b) (4 points) Letting $D = \{x^{2020} + y^{2020} \leq 1\}$, compute the line integral $\int_{\partial D} P\,dx + Q\,dy$ in the counter-clockwise direction.

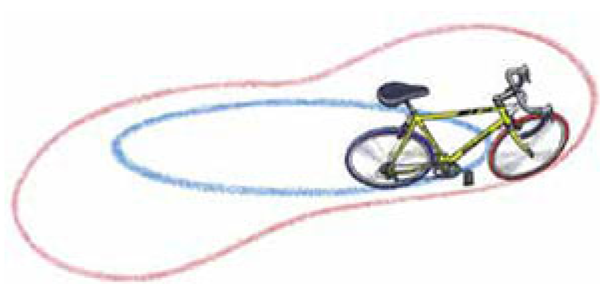
(c) (1 point) Does your calculation in part (b) violate Green's theorem? Explain.

5. (20 points) Suppose you are riding a bike on the plane without slipping. The latter condition of "no slipping" means that if $p \in \gamma$ is a point in the trajectory of the back wheel of the bicycle, then the front wheel lies on the tangent line to γ at point p:



In this problem you will be proving, step-by-step, the following theorem:

Bike theorem. If a bike with a frame of length b made a loop on the plane without slipping, so that the traces of its front and back wheels are disjoint simple closed curves, then the area enclosed between them is independent of the trajectory and equals to πb^2.



(a) (4 points) Prove the theorem for the case when the back wheel's trajectory is a circle.

Hint: use the "no slipping" condition to draw the positions of the front wheel and then determine its trajectory. Then use Pythagoras' theorem to compute the area.

(b) (3 points) Using the (extended) Green's theorem, express the area enclosed between the wheels' trajectories as a difference of line integrals over positively oriented simple closed curves.

Hint: consult the last problem from the week 9 problem set to make an appropriate choice for P and Q.

(c) (3 points) Let C be the trajectory curve of the back wheel, parametrized by $x(t), y(t), 0 \leqslant t \leqslant a$. Let $\theta(t)$ be the angle the bike frame makes with the positive x-axis at time t. Give a parametrization of the trajectory curve of the front wheel C'.

(d) (3 points) Using the parametrization obtained in (c), express the line integral from (b) as a definite integral I with respect to dt. Don't forget to cancel out some of the terms!

(e) (3 points) There are still quite a few terms left in the integral I after (d), but we didn't make a use of the "no slipping condition" yet. Express it as an equation on $x'(t), y'(t), \sin\theta(t), \cos\theta(t)$.

Hint: express the slope of the tangent vector to C using $\theta(t)$.

(f) (2 points) Using the total change of θ along the curve C, find the value of the term in the integral I with coefficient b^2.

Warning: the total change is not zero!

(g) (2 points) Use (e) to express the remaining integrand in I as a derivative of some function. Use the Fundamental Thorem of Calculus to conclude the Bike Theorem.