MTHE224 - Midterm Test
All Sections
Thursday Oct 24, 2024 (6 PM - 8 PM)

Instructions:

- Write your name and student number on this page.
- Read each question carefully.
- No aids except for calculators are permitted.
- There are 5 questions worth 10 points each.
- You only need 40 out of 50 points for full marks on this test.
- For full marks, write clear, consistent and complete solutions.

Do not start the test until instructed to do so by your proctor.

Problem 1

(10pts) A deck of 52 playing cards, containing all 4 aces, is randomly divided into 4 piles of 13 cards each. Define events E_1, E_2, E_3, and E_4 as follows:

$$\begin{cases} E_1 & = \{\text{a pile has A}\spadesuit\}, \\ E_2 & = \{\text{A}\spadesuit\ \text{A}\heartsuit \text{ in different piles}\}, \\ E_3 & = \{\text{A}\spadesuit\ \text{A}\heartsuit\ \text{A}\diamondsuit \text{ in different piles}\}, \\ E_4 & = \{\text{A}\spadesuit\ \text{A}\heartsuit\ \text{A}\diamondsuit\ \text{A}\clubsuit \text{ in different piles}\}. \end{cases} \tag{1}$$

(a) Find the probability $\mathbb{P}(E_2 \mid E_1)$.

(b) Find the probability $\mathbb{P}(E_1 \cap E_2)$.

(c) Find the probability $\mathbb{P}(E_3 \mid E_1 \cap E_2)$.

(d) Find the probability $\mathbb{P}(E_4 \mid E_1 \cap E_2 \cap E_3)$.

(e) Hence, find the probability $\mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4)$.

Continue your solution of Problem 1 here.

Problem 2

(10pts) A gambler has in her pocket a fair coin and a two-headed coin.

(a) She selects one of the coins at random, and when she flips it, it shows heads. Find the probability that the coin is fair.

(b) Suppose she flips the same coin a second time and it shows heads again. Now find the probability that the coin is fair.

Now suppose she also has a weighted coin with probability $\frac{1}{3}$ of flipping heads. Define random variables E_1 and E_2 as follows:

$$\begin{cases} E_1 & = \text{number of flips until heads is flipped} \\ E_2 & = \text{number of flips until two heads are flipped.} \end{cases} \tag{2}$$

(c) Identify the distribution of E_1.

(d) Find the expected value $\mathbb{E}[E_1]$. (*Hint: Condition on the first flip.*)

(e) Hence, find the expected value $\mathbb{E}[E_2]$.

Continue your solution of Problem 2 here.

Problem 3

(10pts) Let the probability density function of X equal

$$f(x) = \begin{cases} ce^{-2x}, \ x \geq 0 \\ 0, \ x < 0. \end{cases} \tag{3}$$

(a) Find $c \in \mathbb{R}$ such that $f(x)$ is a probability density function.

(b) Find $\mathbb{P}(X > 2)$.

(c) Find the expected value $\mathbb{E}[X]$.

(d) Find the variance $\mathbb{V}(X)$.

(e) Find the median m.

(f) Identify the distribution of X.

Continue your solution of Problem 3 here.

Problem 4

(10pts) We want to show that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}. \tag{4}$$

(a) Show that, if $X_1 \overset{d}{\sim} \text{Poi}(\lambda_1)$ and $X_2 \overset{d}{\sim} \text{Poi}(\lambda_2)$, then $X_1 + X_2 \overset{d}{\sim} \text{Poi}(\lambda_1 + \lambda_2)$.

(b) By induction, deduce that, if $X_k \overset{d}{\sim} \text{Poi}(\lambda_k)$, then $\sum_{k=1}^{n} X_k \overset{d}{\sim} \text{Poi}(\sum_{k=1}^{n} \lambda_k)$. (*Hint: Assume it is true for N and show it is also true for $N+1$.*)

(c) Show that, if $S_n \overset{d}{\sim} \text{Poi}(n)$, then

$$\mathbb{P}(S_n \leq n) = e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}. \tag{5}$$

Let $\{X_k\}$ be independent and identically distributed $\sim \text{Poi}(1)$.

(d) Using the central limit theorem, show that

$$\sqrt{n}\left(n^{-1}\sum_{k=1}^{n} X_k - 1\right) \xrightarrow{d} \mathcal{N}(0,1). \tag{6}$$

(e) Deduce that

$$\lim_{n\to\infty} \mathbb{P}(\sqrt{n}\left(n^{-1}\sum_{k=1}^{n} X_k - 1\right) < 0) = \frac{1}{2}. \tag{7}$$

(f) Hence, deduce that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}. \tag{8}$$

Continue your solution of Problem 4 here.

Problem 5

(10pts) We want to calculate the probability that the limit state of a structure is exceeded. To do so, we consider the strength R and the load effect Q as random parameters. The limit state is exceeded when $Q > R$.

Suppose the load effect Q is given by

$$Q = Q_D + Q_L + Q_S, \tag{9}$$

where Q_D, Q_L and Q_S are the dead, live and snow load intensities, respectively. Also suppose the load intensities are normally distributed and satisfy the following table

load	μ	V
Q_D	180	0.10
Q_L	85	0.25
Q_S	170	1.0

where μ is the expected value, σ is the standard deviation and $V := \frac{\sigma}{\mu}$ is the coefficient of variation.

(a) Find the mean of Q.

(b) Find the variance of Q.

Suppose the strength R is given by

$$R = 580R_0, \tag{10}$$

where R_0 is normally distributed with mean 1.11 and coefficient of variation 1.13.

(c) Find the mean of R.

(d) Find the variance of R.

(e) Identify the distribution of $Q - R$.

(f) Hence, calculate the probability that the limit state is exceeded.

Continue your solution of Problem 5 here.

Use this page if additional space is required. Clearly state the question number being answered and refer the marker to this page.

Use this page if additional space is required. Clearly state the question number being answered and refer the marker to this page.

Useful Formulae

$$\mathbb{P}(A) = \frac{|A|}{|S|} \qquad \mathbb{P}(A) + \mathbb{P}(A^c) = 1$$

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \qquad \mathbb{P}(A \sqcup B) = \mathbb{P}(A) + \mathbb{P}(B)$$

$$\mathbb{P}(A \,|\, B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \qquad A \perp\!\!\!\perp B \implies \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

$$\mathbb{P}(B \,|\, A) = \frac{\mathbb{P}(B)\mathbb{P}(A \,|\, B)}{\mathbb{P}(B)\mathbb{P}(A \,|\, B) + \mathbb{P}(B^c)\mathbb{P}(A \,|\, B^c)}$$

$$\mathbb{E}[X] = \mu = \sum x\mathbb{P}(X = x) \qquad \mathbb{E}[X] = \mu = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$\mathbb{E}[\alpha X + \beta Y] = \alpha\mathbb{E}[X] + \beta\mathbb{E}[Y] \qquad \mathbb{E}[X] = \sum \mathbb{E}[X \,|\, Y = j]\mathbb{P}(Y = j)$$

$$\mathbb{V}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \qquad \mathbb{V}(X) = \sigma^2 = \sum (x - \mu)^2\mathbb{P}(X = x)$$

$$\mathbb{V}(X) = \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f_X(x)\,dx \qquad X \perp\!\!\!\perp Y \implies \mathbb{V}(\alpha X + \beta Y) = \alpha^2\mathbb{V}(X) + \beta^2\mathbb{V}(Y)$$

$$X \sim \text{Bin}(n, p) \implies \mathbb{P}(X = j) = C^n_j p^j (1 - p)^{n-j} \qquad \mathbb{E}[X] = np \qquad \mathbb{V}(X) = np(1 - p)$$

$$X \sim \text{Geo}(p) \implies \mathbb{P}(X = j) = p(1 - p)^{j-1}$$

$$X \sim \text{Poi}(\lambda) \implies \mathbb{P}(X = j) = e^{-\lambda}\frac{\lambda^j}{j!} \qquad \mathbb{E}[X] = \lambda \qquad \mathbb{V}(X) = \lambda$$

$$X \sim \mathcal{N}(\mu, \sigma^2) \implies f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad \mathbb{E}[X] = \mu \qquad \mathbb{V}(X) = \sigma^2$$

$$X \geq 0, a > 0 \implies \mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a} \qquad \frac{\sum_1^n X_j}{n} \to \mu \qquad \sqrt{n}\frac{n^{-1}\sum_1^n X_j - \mu}{\sigma} \to \mathcal{N}(0, 1)$$